STRENGTH OF MATERIALS

THIRD EDITION

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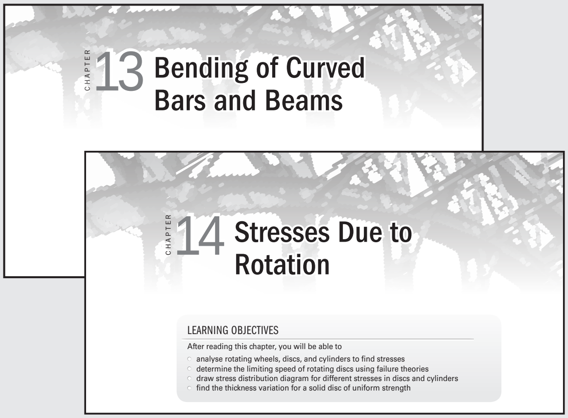


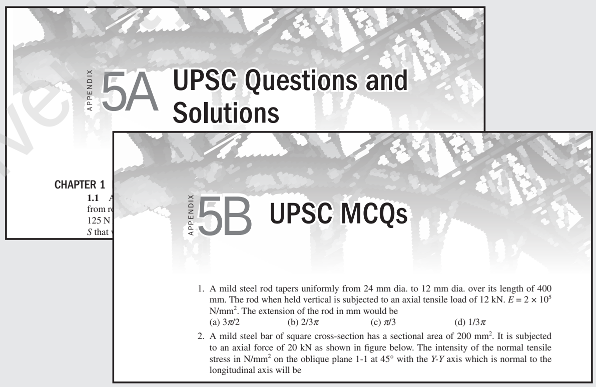
Coverage

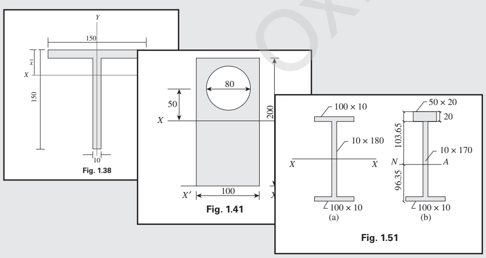
This new edition expands its coverage through two new chapters—Chapter 13: *Bending of Curved Bars and Beams* and Chapter 14: *Stresses due to Rotation.* Addition of new topics such as Bearing Stress, Flitched Beams, Application of Euler's and Rankine's Formulae, etc. enhances the depth of coverage in chapters.

UPSC Questions

Appendix 5 offers a rich resource of solved examples from previous years' UPSC question papers for aspiring candidates of competitive examinations. Problems from the IES, IFS, and UPSC mains civil engineering papers have been solved and placed chapter-wise according to the theoretical concept they are based on.

MCQs from the objective papers of IES examination have also been given in this appendix.



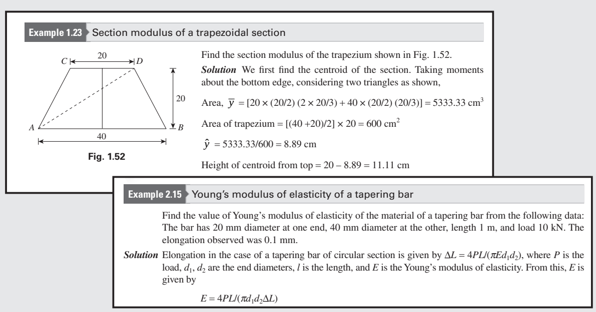


Illustrations

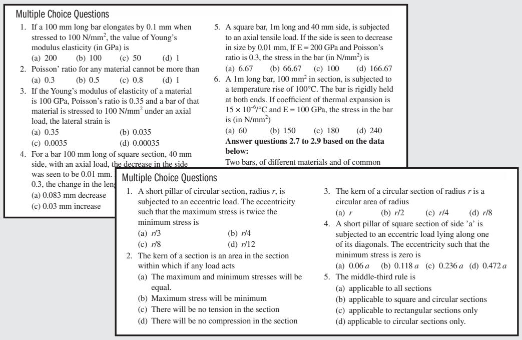
Improved 2D and 3D illustrations for better readability and pictorial representation of objects and structures. Grey-scale shading enhances the figures further to enable a three-dimensional effect.

Application-oriented Solved Examples

Applications of theoretical concepts to different sections and structures in varying load conditions have been explained using solved examples. This is the USP of the book. It ensures clear understanding of the concept and helps formulate analytical thinking in the reader's mind with minimal description and more solved examples.

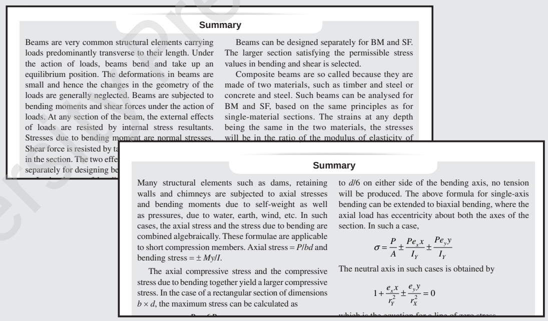


the Book



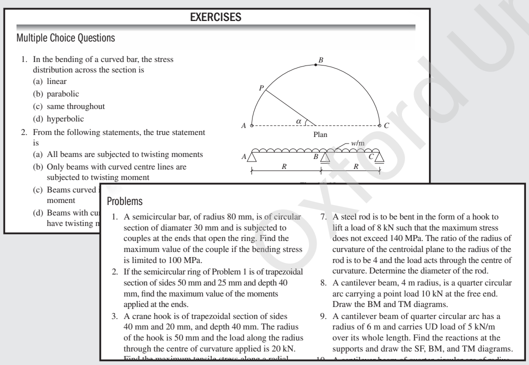
Multiple-Choice Questions

Multiple-choice Questions have been included in the end-of-chapter exercises. This makes it easy for students and faculty to refer to the MCQs as soon as the chapter is completed. Test-generator based MCQs are available to students on our Online Resource Center (india.oup.com/orcs/9780199464739) where they can mark their answers and also evaluate their results online.



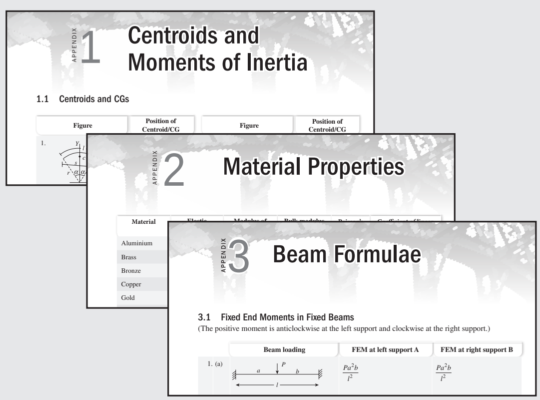
Summary

The summary at the end of every chapter briefly recapitulates the topics covered in the chapter for a quick look-up.



End-of-chapter Exercises

Exercises at the end of chapters include unsolved problems, review questions, and MCQs that assist the student in practising and revising the taught theory. Faculty can use these for the purpose of classroom teaching.



Appendices

A host of appendices with essential data on Centroids and Moments of Inertia, Material Properties, and Beam Formulae provide a ready reference for problem solving to students and faculty alike.

Preface to the Third Edition

It was gratifying to note the enthusiastic response received by the second edition of *Strength of Materials* over the years from the academic fraternity of faculty and students. The basic feature of laying emphasis on the understanding of fundamental concepts and principles has been widely appreciated. Responding to this ever increasing response by the academic fraternity, it gives me great pleasure to present the third and enlarged edition of the book.

New to the Third Edition

The third edition of the book has taken into account the feedback received from many users. Two new chapters are included in addition to expanding the existing chapters by adding new solved examples. To help those students who may appear for many competitive examinations, questions from such examinations, both subjective and objective, with solutions are included.

A brief summary of the key features of the third revision are listed below:

1. Two additional chapters are included to take care of the content followed in some universities. Chapter 13 on 'Bending of Curved Bars and Beams' and Chapter 14 on 'Stresses due to Rotation' have been prepared on the same lines as the other chapters.
2. Additional solved examples have been added in many chapters to facilitate students' understanding of basic concepts and principles.
3. Multiple Choice Questions were available with the second edition as well but were put on the companion website. To facilitate students' preparation for future studies, nearly 200 MCQS are now included as a part of the end-chapter exercise. The key for the MCQs is provided in the appendix.
4. The book has found appreciation for its highly visual nature with more than 1000 illustrations. Keeping this in focus, the quality, visibility, and readability of illustrations have been enhanced throughout the book.
5. Keeping the future requirements of the students in mind, there is a separate section on UPSC examinations, both as conventional questions and objective type questions. The solutions to conventional questions have been provided and a key for the MCQs has been given. The detailed solutions to these MCQs will be available in the companion website for students for reference.
6. Some rearrangement of the chapters has been made to rationalize the presentation of the content in a logical order. The structure of the book is as follows:

Chapter 1 is devoted to a recapitulation of the relevant basic concepts of applied mechanics. It also deals with properties of sections such as moment of inertia (MOI), product of inertia, polar MOI, etc., which constantly find application in structural analysis. A thorough understanding of these topics is necessary for further work.

Chapter 2 deals with elementary concepts of stresses and strains, Hooke's law, elastic constants, stresses in compound sections, temperature stresses and some simple indeterminate problems, and mechanical properties of materials. This chapter is important for further studies in stress analysis.

Chapter 3 deals with an important and common structural element, the beam. The concepts of bending moment (BM) and shear force (SF) are introduced along with the differential relationships and methods of showing variations in BM and SF diagrams. The analysis of statically determinate rigid frames has also been covered.

Chapter 4 follows the concepts presented in Chapter 3 by introducing the bending equation along with methods for calculating bending and shearing stresses followed by their distribution. The concepts of shear centre and unsymmetrical bending have also been covered in this chapter.

Chapter 5 is about the effect of combining direct and bending stresses, and their effects, which are important for analysing structures subjected to lateral pressure, such as chimneys, retaining walls, dams, etc.

Chapter 6 deals with the very important aspect of deformations in beams. The deformations have been calculated using different methods such as double integration, area-moment theorems, and conjugate beam. A thorough grounding of the concepts in this chapter is important for further analysis.

Chapter 7 deals with another important structural element shafts. Torsion and torsional shear have been dealt with in detail.

Chapter 8 provides an analysis of plane stress, both analytically and graphically. The calculation of principal stresses and maximum shear stress has been explained in detail and illustrated with a number of examples.

Chapter 9 deals with strain energy, which has many applications in later studies. Strain energy due to uniaxial, bending, and shear stresses and due to torsion has been explained and illustrated. Suddenly applied and impact loads have also been covered.

Chapter 10 describes another common structural element, the column, which also introduces the concept of stability. Euler's critical load method and the Rankine–Gordon method have been explained in detail.

Chapter 11 deals with some special structural elements such as springs, thin and thick pressure vessels, and elastic theories of failure.

Chapter 12 deals with the truss or pin-jointed plane frame. The method of joints, method of sections, the relevant graphical methods, and the method of tension coefficients have been illustrated.

Determining deflections in trusses using Castigliano's theorem and unit load methods and graphically using Williot-Mohr diagrams are also covered.

Chapter 13 discusses the bending of curved bars and beams. Stresses developed in bars with large curvatures, closed rings and chain links have been dealt with in detail. It also studies various types of beams curved in plan case by case.

Chapter 14 deals with stresses due to rotation in various structural elements such as circular rings, solid discs, hollow and solid cylinders, etc.

Chapter 15 is an introduction to indeterminate structural analysis. The basic concepts of flexibility and stiffness methods have been covered. Simple indeterminate structures such as fixed and continuous beams. It introduces the flexibility method of analysis and illustrates Clapeyron's theorem of three moments in detail.

Chapter 16 is intended as an introduction to advanced structural analysis. The method of moment distribution, credited to Hardy Cross, as applied to continuous beams and frames with no sway as well as side sway, has been introduced in this chapter.

Appendices provide the centroids and moments of inertia of standard structural elements. Characteristic properties of a list of different materials have been given as a ready reference along with a table that lists the beam formulae. Additionally this edition has a separate appendix that contains solved UPSC problems and MCQs from previous 5 years' UPSC papers.

To reinforce the theoretical concepts, the text is supplemented by a large number of worked-out examples with step-by-step solution procedures as well as a large number of review questions and exercise problems. The book also provides useful tables both in the text and in the appendices for ready reference.

I sincerely hope that the users of the book, both faculty and students, will find the third edition very useful for a better understanding of the subject for its application in advanced courses.

Acknowledgements

I wish to gratefully acknowledge the great effort put in by the OUP editorial team in preparing the book in such an elegant and readable format. Any suggestions for improvement of the book are welcome. Any suggestion can be sent to the publishers by logging on to their web site (www.oup.co.in) or by writing to the author at rsmani2k@yahoo.com.

R. Subramanian

Preface to the First Edition

The subject of strength of materials or mechanics of solids involves analytical methods for determining the strength, stiffness, and stability of the various load-carrying structural members. A thorough understanding of the underlying principles is useful to civil engineers and architects, and cuts broadly across all branches of engineering with several applications.

This book has been specially written for undergraduate students of engineering taking a first course on the subject. While there are many books available on this subject, I have written this book with a focus on the concepts and their engineering applications. During my tenure as a teacher, I have observed that a large number of students find it difficult to grasp the concepts and principles, and often omit several topics. I wanted to present the subject matter in an easy-to-comprehend form and explain the concepts and principles by applying them to a large number of examples, which, I hope, will help the students to internalize them. A strong grasp of these concepts and principles will help them to understand advanced topics with ease.

About the Book

I have attempted to make the presentation as lucid and comprehensible as possible. Going from the simple to the complex, the concrete to the abstract, and the known to the unknown are the cardinal tenets followed in arriving at the format and detailing of the book. There is lack of uniformity in the curriculum followed in different universities and engineering colleges. Therefore, during the preparation of the manuscript, I have attempted to give as comprehensive a coverage as possible of the topics, taking the various curricula in effect into consideration.

Acknowledgements

I am grateful to the Bureau of Indian Standards for permission to reproduce a part of the steel tables from their publication. I am also grateful to the editorial team at Oxford University Press for the excellent job done in bringing this book out in a short time with a high degree of accuracy and precision. I am grateful as well to a large number of my teachers and students who have directly or indirectly helped me in this endeavour.

Despite best efforts by all concerned, it is possible that some errors might have crept into the final form in your hand. It will be appreciated if such errors are pointed out to me or to the publishers. In addition, suggestions from teachers and students regarding any additions in topics or subtopics required to make the book more useful are welcome. Such comments or suggestions will be taken care of in subsequent editions of the book.

R. Subramanian

Brief Contents

Detailed Contents

CHAPTER 1 Basic Concepts

LEARNING OBJECTIVES

After reading this chapter, you will be able to

- recall and apply relevant basic concepts from mechanics for solving problems,
- appreciate the importance of such basic concepts in engineering analysis,
- define the terms moment of inertia and second moment of area,
- state and explain the parallel and perpendicular axes theorems,
- compute the second moment of area of a given section,
- calculate the radius of gyration of a given section,
- calculate the product of inertia of a given section,
- calculate the principal second moment of area of unsymmetrical sections and the directions of principal axes of inertia, and
- compute the section modulus of a given section.

1.1 INTRODUCTION

The methods and techniques used in the analysis of structures or machine elements are based upon the concepts and principles, which the reader would have studied in a first course in engineering mechanics. This chapter starts with a recall of relevant concepts and principles from that subject. This chapter also deals with some basic concepts such as moment of inertia, product of inertia, and section modulus, which find applications in many structural or machine elements to be dealt with in later chapters.

1.2 BASIC PRINCIPLES OF MECHANICS

Before taking up the principles, definition of the following terms must be clearly understood:

Matter is any substance that occupies space.

Particle in solid geometry is analogous to a point in plane geometry. A particle has mass but has no dimensions.

Body is matter that is bounded by a closed surface. Bodies can be classified as rigid and deformable.

A *rigid body* is a body that does not undergo any deformation, change in shape and size, on application of a force. All bodies in nature are deformable on application of a sufficiently large force. We would not have been able to fabricate many things that we use in day-to-day life if bodies were not deformable. But for the purpose of certain analysis, as in statics detailed in the next section, bodies are assumed to be rigid (strictly, we assume that deformations are too small that we can consider the body to be rigid).

A *deformable body* is one that undergoes deformations, change in size and shape, on application of a force. Materials have different degrees of deformations on application of forces. Rubber, for example, undergoes deformations on application of a small force. Steel, on the other hand, requires a very large amount of force to deform it. It is necessary to compute the deformations of bodies and it is done in many forms of analysis (see Chapter 3 on Simple Stresses and Strains and Chapter 7 on Deformations in Beams).

Inertia is an inherent property of matter by which it resists any change in its state.

Mass is a quantitative measure of inertia. Bodies undergo different degrees of deformations under the action of forces. Thus, steel has more inertia (mass) than aluminium as, under the action of the same force on two identical bodies of steel and aluminium, the aluminium body will undergo more change in motion than steel.

Space is a region that extends in all directions and contains all bodies. Position of a body in space is located by arbitrarily fixing reference axes and measuring its coordinates with respect to these axes. Rectangular coordinates use three mutually perpendicular axes while cylindrical coordinates use distances and angles to locate position in space.

Time is a measure of duration between successive events occurring. This is important in mechanics as bodies in motion change their position with time.

Equilibrium is a state of rest of a body. State of rest can be defined as a state in which the body does not change its position. Equilibrium equations also cover the case of a body moving with uniform velocity along a straight line.

Motion is a change of position of a body with respect to time. When an unbalanced force acts on a body, it has motion.

Scalar and vector quantities are physical quantities like mass, time, speed, velocity, force, etc. A scalar quantity is one that has only a magnitude as an attribute. Two scalar quantities can be added using the conventional method of addition. Mass, time, length, etc. are scalar quantities. Vector quantities, on the other hand, have a direction in addition to magnitude. They cannot be added like scalar quantities and are manipulated using a different mathematical tool known as *vector algebra*. Force, moment, velocity, etc. are vector quantities.

The following basic principles of mechanics form the foundation on which the entire classical mechanics is based. They are in general proved by experience than mathematically.

1. ***Newton's laws of motion*** There are three Newton's laws of motion, which govern the motion of bodies under the action of forces.
   Newton's first law states that a particle or rigid body will remain in a state of rest or continue to move in a straight line with uniform velocity unless acted upon by an unbalanced force.
   Newton's second law states that a particle's rate of change of momentum is equal to the unbalanced force acting on it and takes place in the direction of the force.
   Newton's third law states that to every action there is an equal and opposite reaction.
2. ***Parallelogram law*** This law states that if two vector quantities are represented by two adjacent sides of a parallelogram, their sum is given by the diagonal of the parallelogram passing through the point of intersection of the two forces. This is discussed in detail in the next section.
3. ***Principle of transmissibility*** This principle states that the effect of a force acting on a rigid body does not change if the force is moved along its line of action to another point on the body.
4. ***Principle of superposition*** This principle states that if a number of forces act on a body, the total effect of all the forces is the summation of the effects of the individual forces. Thus,

if a body is subjected to a number of forces, F_1, F_2, F_3,…, the motion imparted to the body is the summation of the motion imparted to the body by the individual forces F_1, F_2, F_3, … . The principle of superposition also means that the effect of a set of forces acting on a body does not change if another system of forces in equilibrium is added to, or subtracted from, it.

5. Two forces, acting on a body and keeping it in equilibrium, must be collinear, equal, and opposite.
6. ***Newton's Law of Gravitation*** It states that two bodies attract each other by a force given by Gm_1m_2/r^2, where G is the universal constant of gravitation, m_1, m_2 are the masses of the bodies, and r is the distance between them. Based on the law of gravitation, the earth attracts all material bodies with mass. The weight of a body is due to this attraction. Earth's gravity gives to a body of mass m, a weight of gm, where $g = 9.81$ m/s^2 and $g = GM/R^2$, where G = Newton's universal gravitational constant, M is the mass of the earth, and R is the radius of the earth.
7. ***Law of conservation of energy*** It states that energy can neither be created nor destroyed but can change from one body to another or change form. When a force acts on a mass and the body moves, it does work and the work is converted to kinetic energy of the body. Work and energy are scalar quantities and are easier to work with. Energy principles are used in many forms of analysis.

1.3 STATICS

Statics is a branch of mechanics that deals with forces and moments which are in equilibrium. The principles of statics are also applicable to bodies which are in motion but without acceleration.

1.3.1 Force

Force is a vector quantity and may be represented as shown in Fig. 1.1(a). The three attributes of force are shown here. AB shows the direction of the force, the arrow shows the sense (acting from A to B), and B is the point of application. The line ab show the direction of the force and the arrow at the end shows the sense of the force as from a tob or $\overrightarrow{ab}$, and the point of application of the force is either at the beginning or at the end of line ab.

Figure 1.1(b) shows a vectorial representation of the force $\overrightarrow{ab}$. In addition to the three attributes discussed above, the vectorial or graphic representation also includes the magnitude of the force represented by line ab to some scale. This representation is required for a graphic solution to problems.

In the SI (the abbreviation of the French form of The International System of Units) units, the unit of force is the newton. One newton is the force required to be applied to a mass of 1 kg to give it an acceleration of 1 m s^{-2}. Since force = mass × acceleration, force (newton) = kg m s^{-2}. Many multiples of this unit are used, e.g. 1 kilonewton = 1000 N, 1 meganewton (MN) = 1000 kN, 1 giganewton (GN) = 1000 MN, etc.

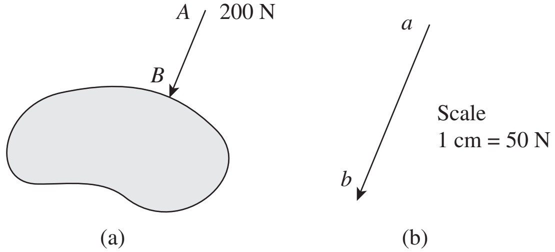
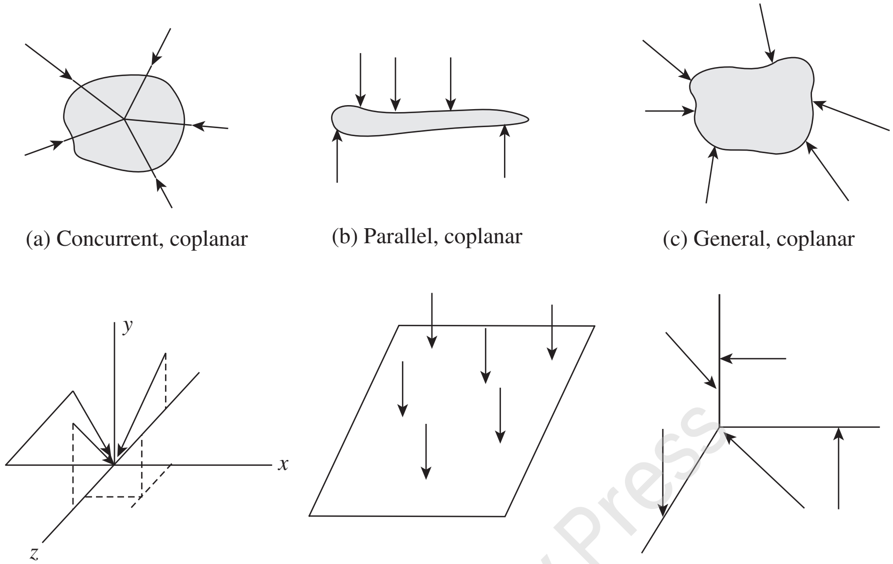


Fig. 1.1

Force systems A number of forces acting on a body forms a force system. Force systems may be coplanar, in which case the lines of action of all the forces of the system lie in a plane. Such systems may also be spatial, when the lines of action do not lie in a plane (Fig. 1.2). Force systems can further be classified as concurrent, parallel and non-concurrent, non-parallel, as shown in Fig. 1.2.



(a) Concurrent, coplanar (b) Parallel, coplanar (c) General, coplanar

(d) Concurrent, space (e) Parallel, space (f) General, space

Fig. 1.2 Force systems

Composition of forces Two given forces acting (concurrently) at a point can be combined into a single force, known as their resultant and having the same effect on the body as the two forces, by the parallelogram law or triangle law of forces. This is shown in Fig. 1.3, where F_1 and F_2 are the two forces and R is their resultant. Analytically, $R^2 = F_1^2 + F_2^2 - 2F_1F_2\cos\theta$, where θ is the angle between the lines of action of the forces.

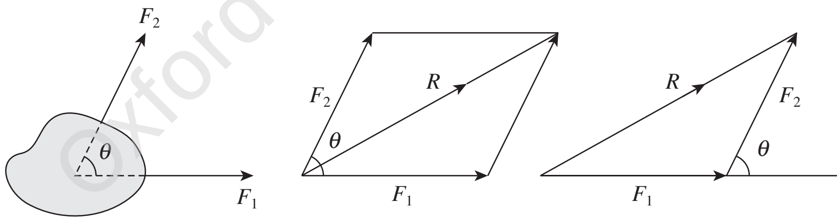


Fig. 1.3 Addition of two vectors

If more than two forces act at a point, the resultant is determined from the polygon of forces which is obtained by the repeated application of the triangle law of forces (Fig. 1.4). The magnitude, line of action, and sense of the resultant are obtained from the force polygon, and the resultant passes through the point of concurrency.

To determine the resultant of more than two forces analytically, we use a combination of resolution and composition. In the concurrent force system shown in Fig. 1.5(a), any of the forces, say F_1, can be resolved into components along perpendicular directions X and Y as $F_1\cos\theta_1$ along X and $F_1\sin\theta_1$ along Y. These are known as rectangular components as they act along mutually perpendicular directions. If all the forces of the system are thus resolved into components, then we have two forces ΣX and ΣY given by

$$\Sigma X = F_1\cos\theta_1 + F_2\cos\theta_2 + F_3\cos\theta_3 + F_4\cos\theta_4$$

$$\Sigma Y = F_1\sin\theta_1 + F_2\sin\theta_2 + F_3\sin\theta_3 + F_4\sin\theta_4$$

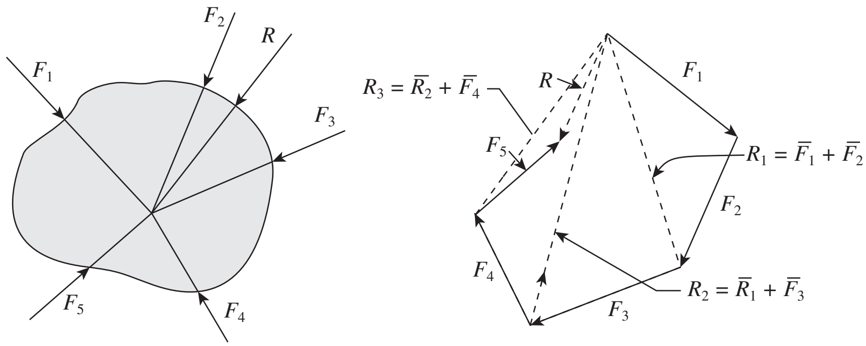


Fig. 1.4

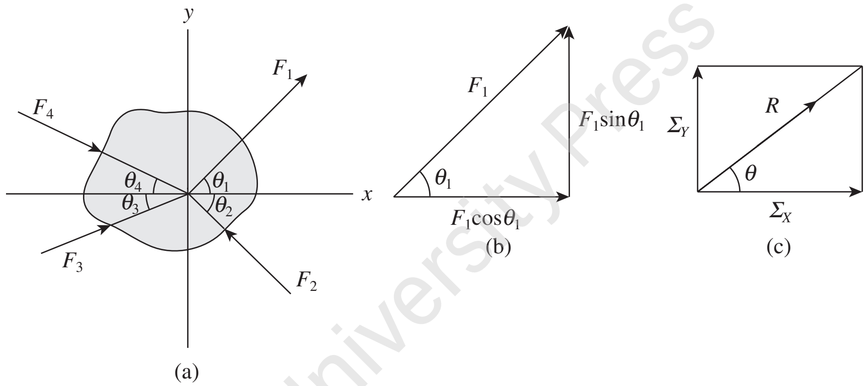


Fig. 1.5

These two forces can be combined into a single force, which is the resultant, whose magnitude can be obtained from

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2$$

The direction of R is obtained as $\theta = \tan^{-1}(\Sigma Y/\Sigma X)$, where θ is the angle made by R with the X-axis.

Moment of a force Moment is a vector quantity like force. As in Fig. 1.6, moment = force × distance = Fd about moment centre O which means moment about an axis passing through O

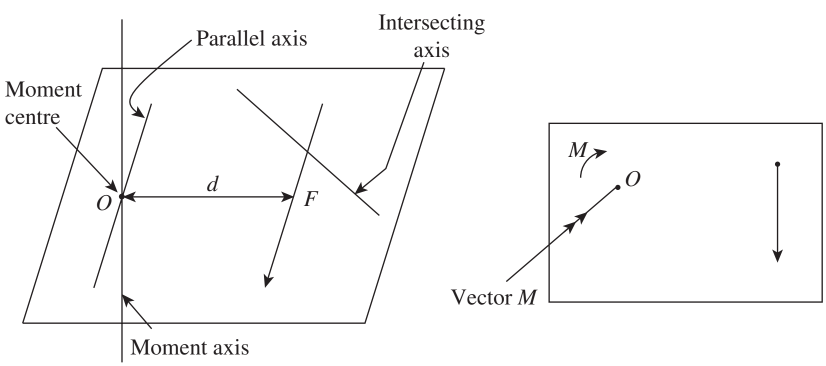


Fig. 1.6

and perpendicular to the plane containing the force. Moments are represented in diagrams using curved arrows about the moment centre, or by a line with double arrows using the right-hand screw notation.

Moment is a measure of the rotating effect of a force. The units of moment are the newton metre (Nm), kilonewton metre (kNm), etc. A force has no moment about an axis parallel to its line of action or intersecting it, as shown in Fig. 1.6.

Varignon's theorem or principle of moments states that *the algebraic sum of the moments of forces is equal to the moment of the resultant of the forces about the same axis*. Moments are given signs according to their nature or direction of rotation. Thus, the moment due to force F_1 is opposed to that due to F_2 about the moment centre O in Fig. 1.7.

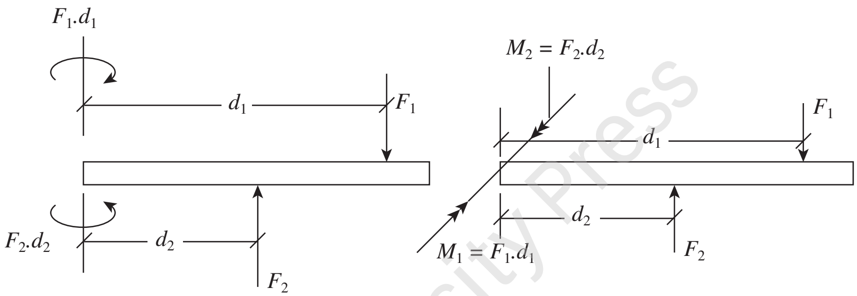


Fig. 1.7

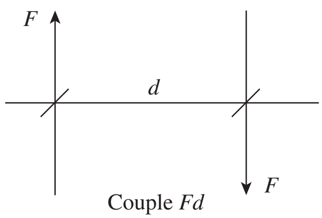
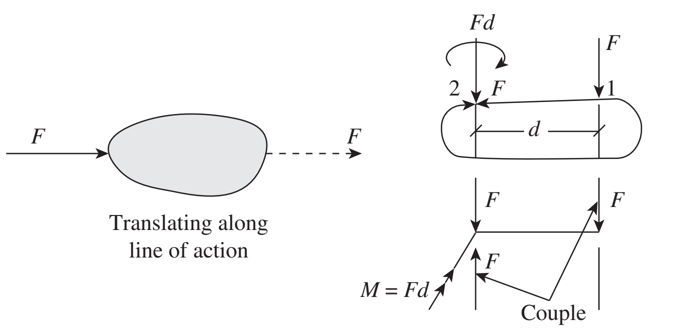


Fig. 1.8

A couple is a special case of a moment due to two equal and opposite forces acting at a distance (Fig. 1.8). The couple has the same moment Fd about any point in its plane.

In rigid body mechanics, a force can be moved along its line of action without altering its effect. But a force can be translated parallel to its line of action only by adding a couple. Let us consider the case of the force F acting at point 1 to be translated to point 2 [Fig. 1.9(b)].

We add equal, opposite and collinear forces F at 2. Note that the addition of these forces do not affect the system of forces and their effect. Considering the given force F at 1 and opposite force F at 2, we have a couple of moment Fd. These two forces can be replaced by the couple Fd. The resultant system is shown in Fig. 1.9(b). The force F has been translated parallel to its line of action, which needs the addition of a couple Fd at 2.



(a) (b) Translating a force parallel to its line of action

Fig. 1.9

The resultant of non-concurrent force systems In the case of non-concurrent forces, the resultant magnitude, direction, and sense can be obtained by resolution and composition or by the method of polygon of forces. But the location of the resultant in space is obtained by the principle of moments or by graphical method of the funicular polygon.

For the parallel force system shown in Fig. 1.10(a), resultant $R = \Sigma F$ = algebraic sum of the forces; the direction of the resultant is the same as that of the given forces and the sense is determined by the sign of ΣF. If the resultant is acting at a distance x from point 1 [Fig. 1.10(b)], then from the principle of moments, $Rx = F_2x_1 + F_3x_2 + F_4x_3 + F_5x_4$, with appropriate signs. From this equation, x can be calculated to locate the resultant.

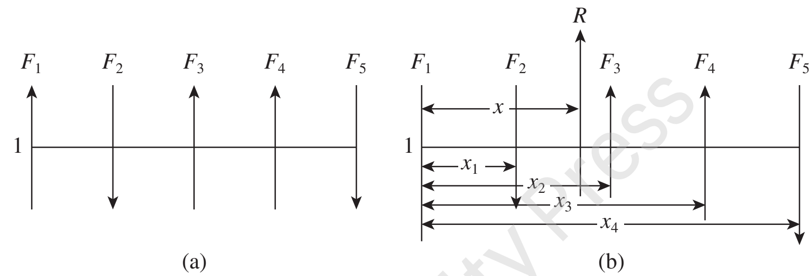


Fig. 1.10

For a general coplanar force system shown in Fig. 1.11(a), the same principles apply: $R^2 = (\Sigma Fx)^2 + (\Sigma Fy)^2$ gives the magnitude, $\theta = \tan^{-1} (\Sigma Fy/\Sigma Fx)$ gives the direction (and the sense is known from the signs of ΣFy and ΣFx), and its location can be determined from the principle of moments [Fig. 1.11(b)]. $Rd = F_1x_1 + F_2x_2 + F_3x_3 + ...$, where x_1, x_2, x_3, ... are perpendicular distances of the forces rom the moment centre O.

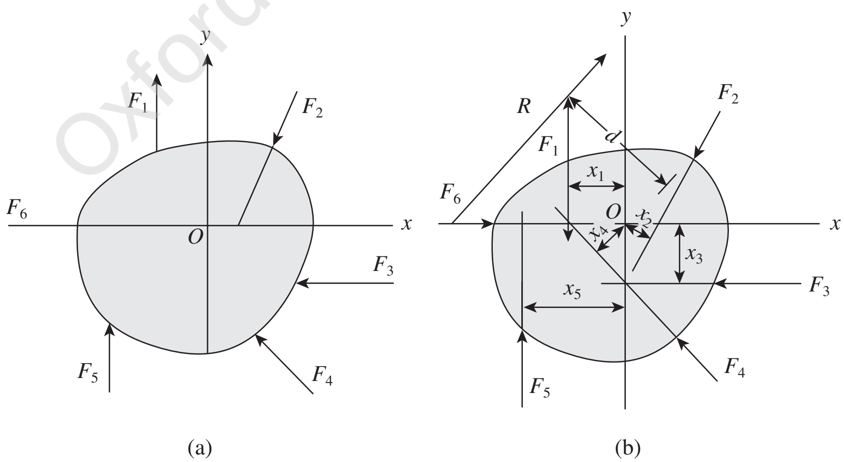


Fig. 1.11

Graphically, taking the force system shown in Fig. 1.12(a), the force polygon is drawn to a force scale in Fig. 1.12(b). Figure 1.12(a) is a space diagram which must be drawn to a linear scale for the graphical solution $\overline{ae}$ is the resultant in magnitude, direction, and sense, in Fig. 1.12(b). In the space diagram, forces are marked using Bow's notation. The letters A, B, etc. are assigned to spaces on either side of the force. Thus, AB is the force F_1, BC is the force F_2, etc.

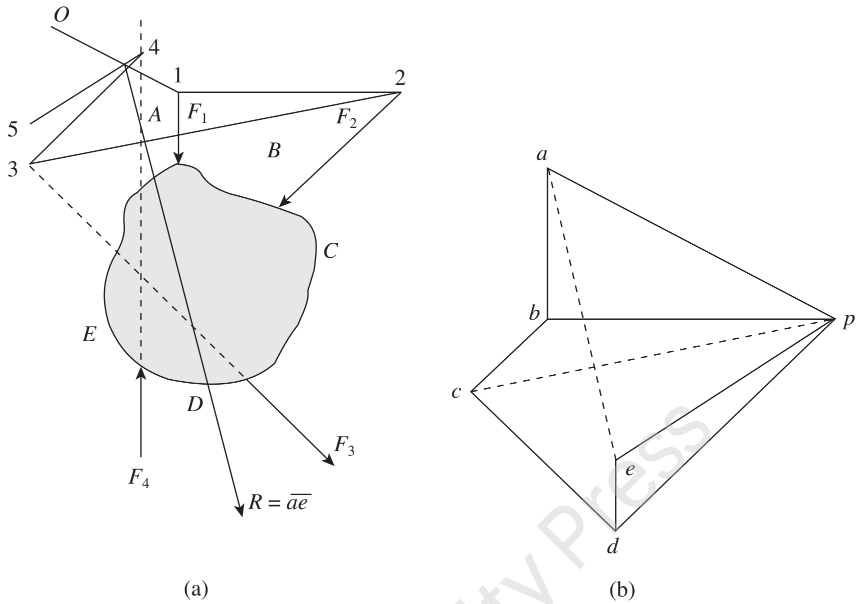


Fig. 1.12

We select a pole p (arbitrarily) and join pa, pb, etc. Draw lines parallel to pa, pb, etc. in the respective spaces A, B, etc. Note that marking of forces by Bow's notation in the space diagram is very important for drawing the funicular polygon. Where lines $\overline{01}$ and $\overline{45}$ of the funicular polygon meet is a point in the line of action of the resultant. The resultant R is shown in the space diagram.

It must be noted that while force polygon is the graphical equivalent of the equation $R^2 = (\Sigma Fx)^2 + (\Sigma Fy)^2$, the funicular polygon is the equivalent of the principle of moments. In principle, the funicular polygon is an ingenious way of resolution of forces such that only to forces (represented by $\overline{01}$ and $\overline{45}$) remain. Their intersection gives a point on the line of action of the resultant.

If the first and last lines of funicular polygon are parallel (Fig. 1.13), this means that the resultant reduces to a couple. The magnitude of the resultant couple can be obtained as (pa × load scale P) × (d × space scale S).

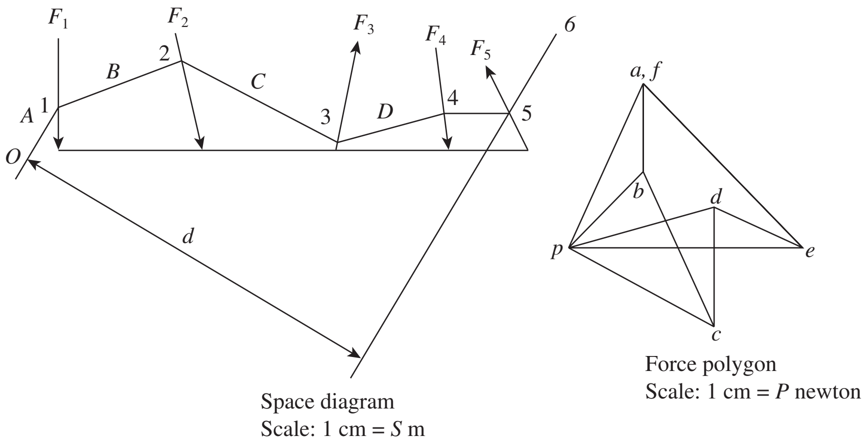


Fig. 1.13 Resultant is a couple

If the force polygon closes, the resultant force $R = 0$. The system may in such a case reduce to a couple as given above. The couple will be zero only when the first and last lines of the funicular polygon are collinear (Fig. 1.14). Here $R = 0$ and $M = 0$.

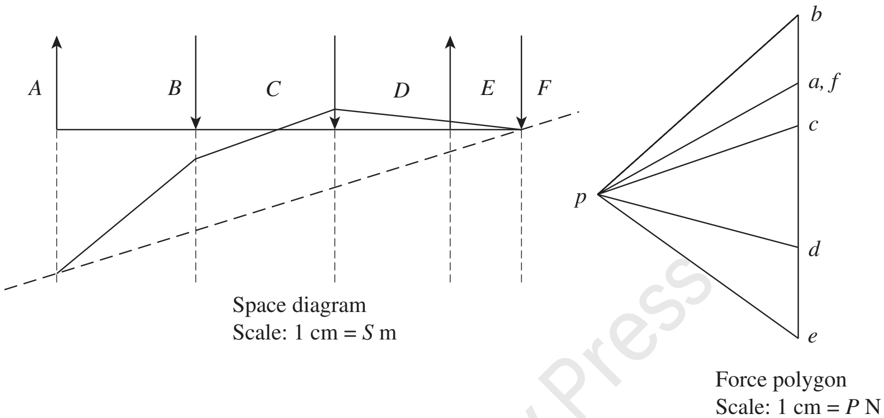


Fig. 1.14 Resultant force and couple zero

1.4 EQUILIBRIUM

Equilibrium means a state of rest or motion without acceleration. Considering our discussions of coplanar force systems, a body subjected to a general coplanar force system may undergo translation, rotation, or both. As shown in Fig. 1.15, the possible displacements of a body are (i) translation Δd in the direction of the resultant force or in terms of its components Δx along the X-axis and Δy along Y-axis and (ii) a rotation in the plane due to a resultant couple.

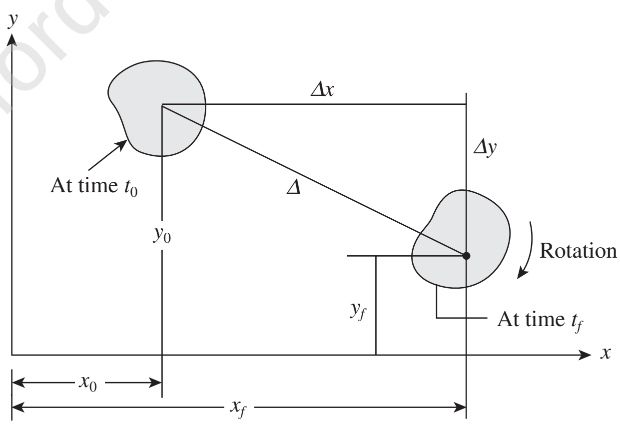


Fig. 1.15

In the state of rest, all these displacements are zero. When the body is not in equilibrium, it continuously changes its position with reference to the X- and Y-axes.

1.4.1 Conditions of Equilibrium

Depending upon the force system acting on the body and the condition that translation and rotation are zero in equilibrium, the following conditions of equilibrium can be derived.

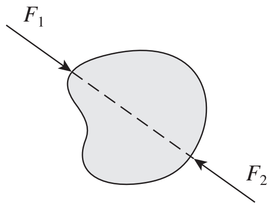


Fig. 1.16

Two-force system When a body acted upon by only two forces is in equilibrium, these forces must be equal, opposite, and collinear (Fig. 1.16).

Three-force system If a body is in equilibrium under the action of three non-parallel forces, the condition of equilibrium is that the forces should be coplanar and concurrent (Fig. 1.17). This can be easily verified. If S is the resultant of any two of the forces, say P and Q, in Fig. 1.17(b), the body is now acted upon by only two forces S and R. S and R should be opposite, collinear, and equal. Since S passes through the intersection of P and Q, R should pass through the same point. P, Q, and R are, therefore, concurrent. S lies in the same plane as P and Q, R should lie in the same plane. P, Q, and R, therefore, should be coplanar and concurrent for equilibrium.

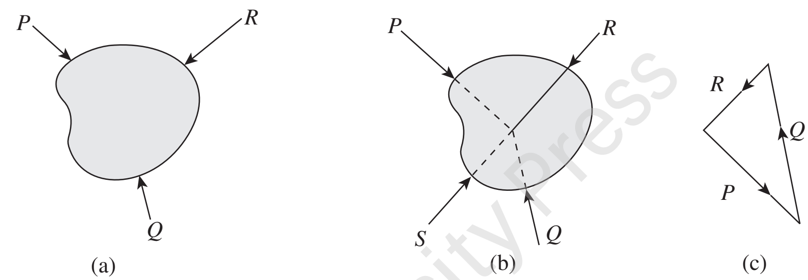


Fig. 1.17

Graphically, the forces P, Q, and R should form a closed triangle as shown in Fig. 1.17(c) for equilibrium.

Concurrent, coplanar force system In the case of a concurrent, coplanar force system, the possible resultant is a force passing through the point of concurrency O (Fig. 1.18). The possible

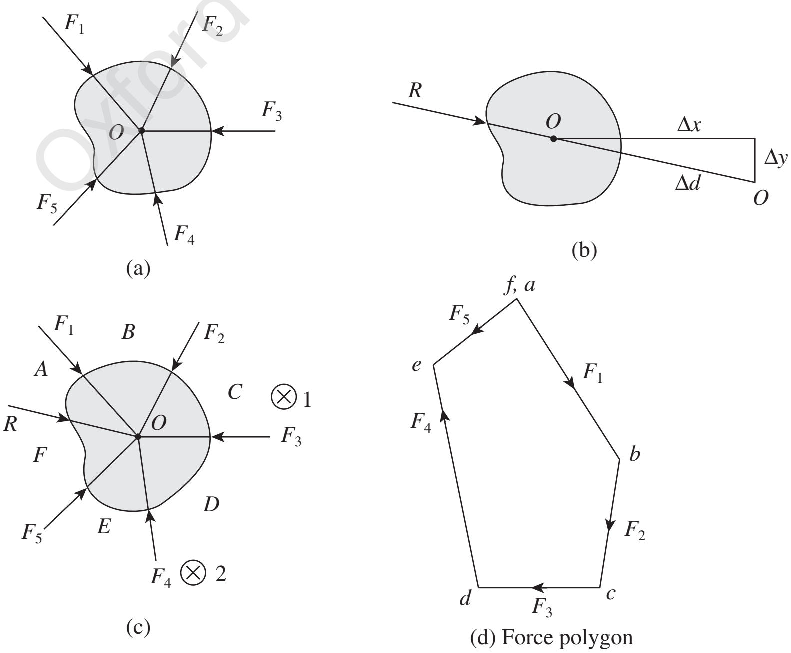


Fig. 1.18

displacement is a translation in the direction of the resultant. This translation Δd in time t is the vector sum of Δx and Δy, the component displacements parallel to x and 11 y. When such a system is in equilibrium, for the displacement to be zero, either $R = 0$ or $\Sigma Fx = 0$ and $\Sigma Fy = 0$. The equilibrium conditions can also be stated in terms of moments as $\Sigma M_1 = 0$ and $\Sigma M_2 = 0$, where ΣM_1 and ΣM_2 are algebraic sums of moments about points 1 and 2, and points 1 and 2 are not collinear with the point of concurrency O [Fig. 1.18(c)]. This can be easily verified. If $\Sigma M_1 =$ 0, either $R = 0$ or R passes through point 1. To eliminate the latter possibility, the moment about a second point is taken. If $\Sigma M_2 = 0$, then $R = 0$ provided 1 and 2 are not collinear with O.

Graphical condition of equilibrium The graphical condition for a concurrent, coplanar force system to be in equilibrium is that the force polygon drawn for such a force system closes. The first and last points coincide, which means that $R = 0$ [Fig. 1.18(d)].

Coplanar, parallel force system In the case of the parallel force system shown in Fig. 1.19(a), the resultant force, if any, is parallel to the given forces. Even if $R = 0$, the system may reduce to a couple M. The conditions of equilibrium for such a force system are that $R = 0$ and $\Sigma M = 0$ about any point in the plane of the forces. This ensures that the body neither translates nor rotates.

The conditions of equilibrium can also be stated in terms of moments as $\Sigma M_1 = 0$ and $\Sigma M_2 = 0$, where 1 and 2 are moment centres such that the line joining 1 and 2 is not parallel to the lines of action of forces. $\Sigma M_1 = 0$ ensures that there is no resultant couple and the resultant, if any, passes through point 1. $\Sigma M_2 = 0$ means that $R = 0$ since R cannot pass through points 1 and 2 [Fig. 1.19(b)].

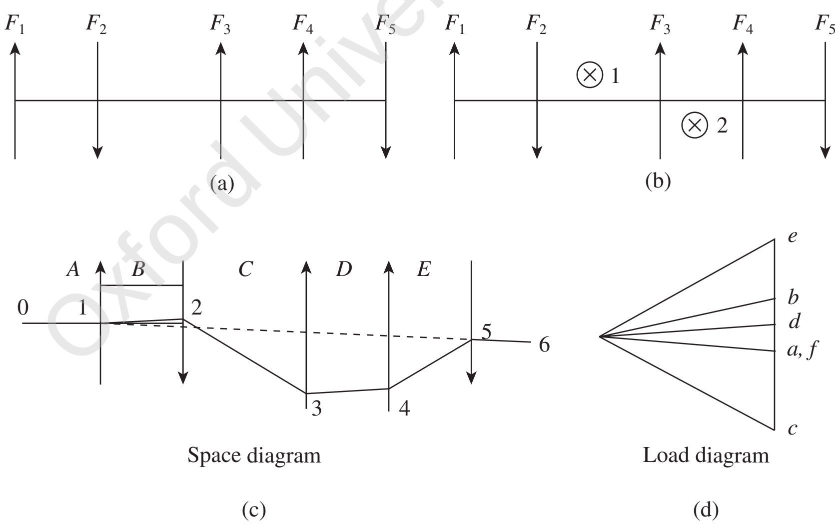


Fig. 1.19

Graphical conditions of equilibrium $R = 0$ requires that the force polygon closes. In the case of a parallel force system, the force polygon degenerates into a straight line as in Fig. 1.19(d) and closing of the polygon means that the first and last points a and f coincide. To ensure that there is no resultant couple, i.e., $\Sigma M = 0$, the funicular polygon should close. This means that the first and last lines of the funicular polygon are collinear. This is illustrated in Fig. 1.19(c), where the first line 0-1 and the last line 5-6 of the funicular polygon are collinear. The graphical conditions of equilibrium for a parallel force system are (i) the force polygon should close ($R = 0$) and (ii) the funicular polygon should close ($M = 0$).

General coplanar force system A general coplanar force system consists of forces which are neither concurrent nor parallel, as shown in Fig. 1.20(a). If a body acted upon by such a force system is in equilibrium, both linear displacement and rotation of the body are zero. The conditions of equilibrium are, therefore, $R = 0$ and $\Sigma M = 0$. The condition $R = 0$ can be expressed in terms of the rectangular components $\Sigma Fx = 0$ and $\Sigma Fy = 0$.

The equilibrium conditions can be expressed in terms of moments in many ways, as given below.

(i) $\Sigma Fx = 0$, $\Sigma M_1 = 0$, $\Sigma M_2 = 0$ with the condition that the line joining 1 and 2 is not parallel to the Y-axis, or

(ii) $\Sigma Fy = 0$, $\Sigma M_3 = 0$, $\Sigma M_4 = 0$ with the condition that the line joining 3 and 4 is not parallel to the X-axis, or

(iii) $\Sigma M_5 = 0$, ΣM_6, and $\Sigma M_7 = 0$ with the condition that 5, 6, and 7 are not collinear.

These conditions are shown in Fig. 1.20(b), and can be easily proved on arguments similar to the ones described for previous cases.

Graphical condition of equilibrium These conditions can be derived from the above. $R = 0$ requires that the forces polygon, *abcde*, closes (the first and last points coincide). $\Sigma M = 0$ requires that the funicular polygon closes, i.e. the first and last lines of the polygon are collinear.

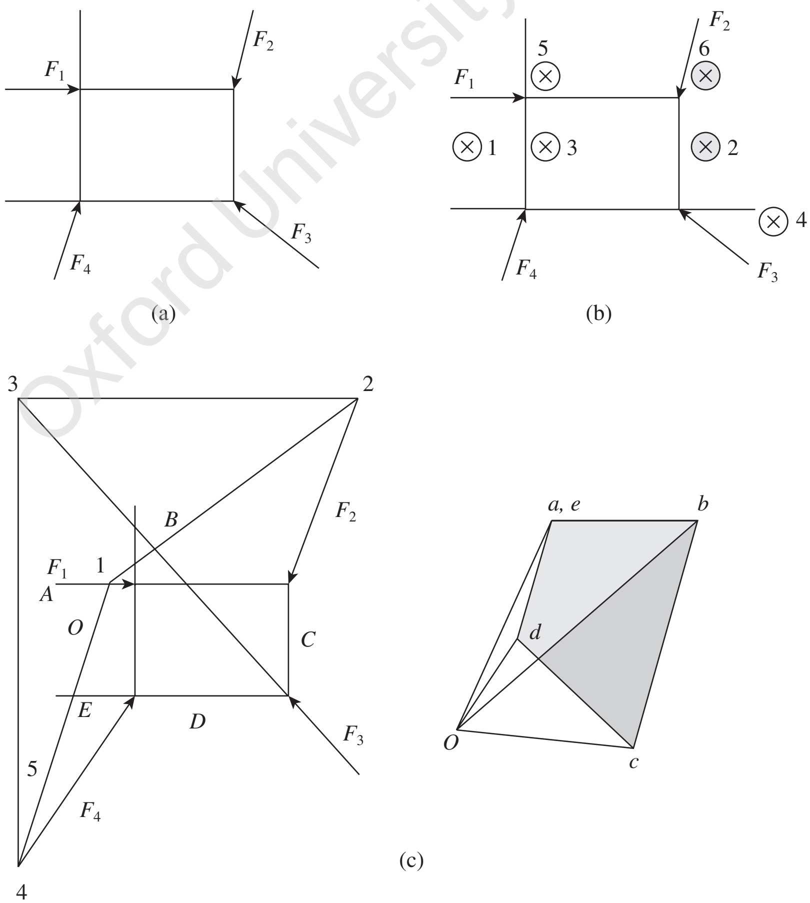


Fig. 1.20

These conditions are shown in Fig. 1.20(c). The first line 0-1 and the last line 4-5 of the funicular polygon are collinear.

We have seen the conditions of equilibrium for coplanar force systems. In the case of spatial force systems, a third dimension z (in addition to x and y) will be added. Thus, the resultant force $R = 0$ means $\Sigma Fx = 0$, $\Sigma Fy = 0$, and $\Sigma Fz = 0$. The moment conditions will be in terms of the moments about the three coordinate axes, as $\Sigma M_x = 0$, $\Sigma M_y = 0$, $\Sigma M_z = 0$, as all the forces do not lie in a single plane.

The conditions of equilibrium given above are used to determine unknown forces in a given system if it is known that the system is in equilibrium.

1.5 BODY CONSTRAINTS AND FREE BODY DIAGRAMS

We have seen analytical and graphical solutions to problems involving forces. The problems solved were direct, i.e., a mathematical model was presented for solution. In practice, problems in structural mechanics are quite complex, and the preparation of the mathematical model itself is quite difficult. To visualize a physical problem, to make suitable assumptions to simplify it, and then to prepare a mathematical model is the first step in structural analysis. The concepts of body constraints and free body diagrams are fundamental to this first step.

1.5.1 Body Constraints

A body constraint is a contrived support or force provided such that the body is in equilibrium. The constraints provide either reactive forces, couples, or both, depending upon the type of support.

Figure 1.21 shows the types of constraints generally assumed. While in a given situation, one may not exactly find parallel physical supports, one can carefully choose from those given in Fig. 1.21 the one that matches the most the physical condition obtaining. Let us briefly discuss the types of constraints that we come across.

Smooth surface It is difficult to find a surface which is perfectly smooth. A smooth surface prevents motion perpendicular to its surface but allows a translation parallel to the surface and rotation. Such a surface thus provides one reactive force pepnicular to the surface.

Rough surface Frictional forces tangential to the surface come into play whenever a body tends to move relative to a rough surface. Such a surface, thus, prevents motion perpendicular to it and also tangential relative motion due to friction. It provides horizontal and vertical reactive forces. One can also say that it provides one reactive force in any direction.

A string or cable tied to the body This is a common type of constraint. A cable provides a reactive force along its direction. The limitation of the cable is that it is effective only when stretched. Otherwise it remains slack, and does not provide a force in the opposite direction.

Roller support The roller support has rollers, which can move over a firm surface, and is attached to the body with a pin. By its very nature, the body can rotate about the pin, and also translate parallel to the surface on which the roller rests. However, it cannot move in a direction normal to this surface. The roller, thus, provides one reactive force perpendicular to the surface on which the rollers rest.

Hinged or pinned support This type of support is firmly attached to the ground and connected to the body through a pin connection. Thus, the body has the freedom to rotate but cannot translate horizontally and vertically or in any direction. The hinged support thus provides one reactive force in any direction or two reactive forces horizontally and vertically.

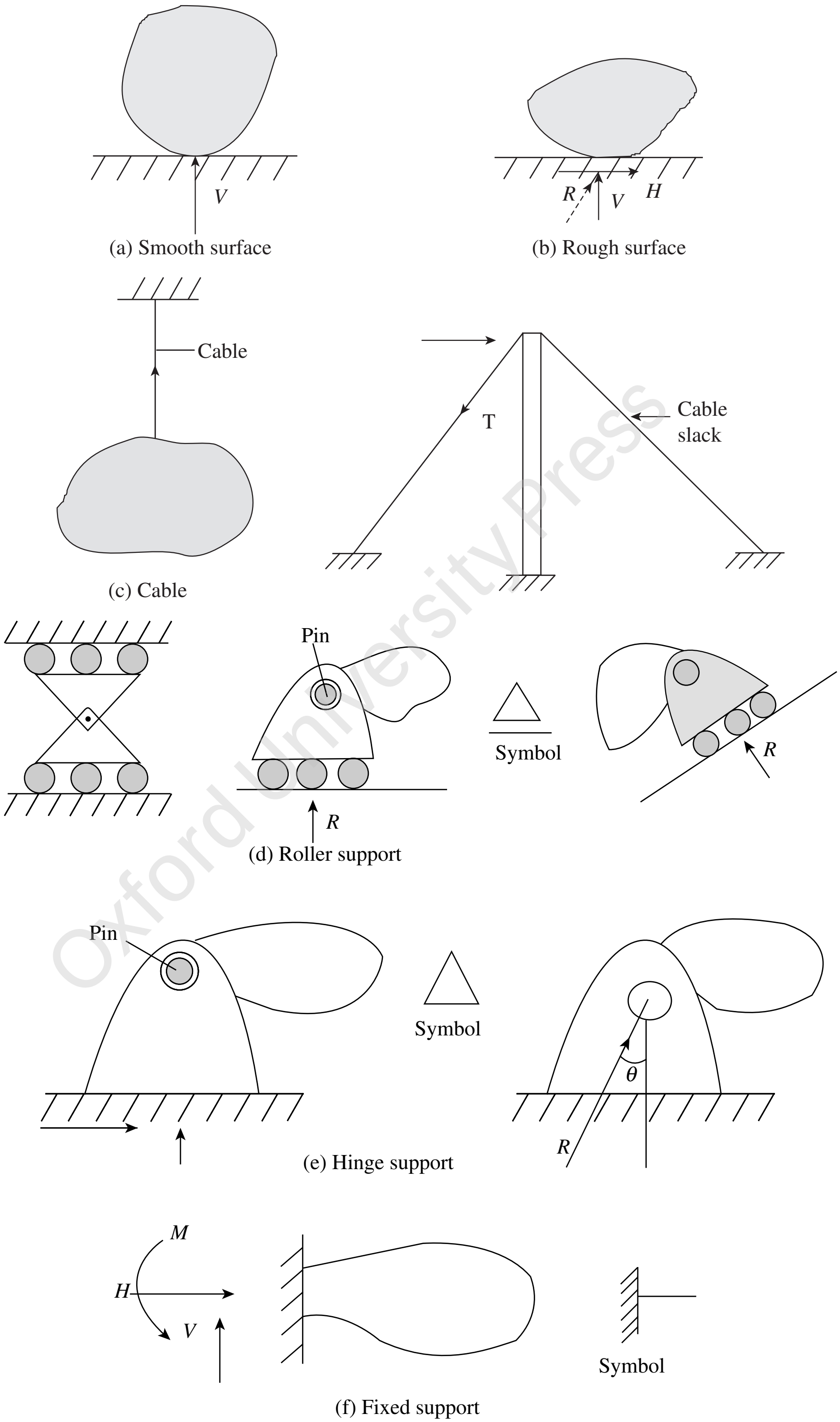
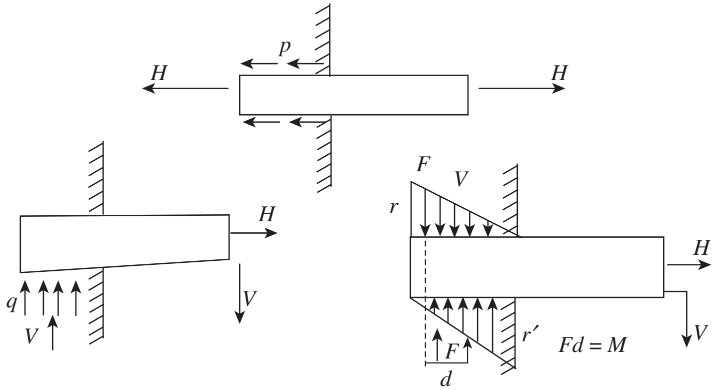


Fig. 1.21 (Contd.)



(g) Reactions at fixed support

Fig. 1.21

Fixed support This type of support does not allow the body to translate or rotate. The reactive forces provided by this support are shown in Fig. 1.21(g). Forces *p* provide the horizontal reactive force, forces *q* provide the vertical reactive forces, and forces *r* and *r″* provide a reactive couple. Such a support alone completely restrains a body acted upon by a general coplanar force system.

The equilibrium conditions can be used to determine the reactive forces provided by the constraints of a body. Depending upon the physical situation and nature of the constraints provided, the appropriate number of reactive forces or couples should be applied to the body.

1.5.2 Free Body Diagram

A number of external or active forces generally act upon a body in equilibrium. So do the reactive forces, couples, or both, provided by the body constraints. The equilibrium conditions are used to evaluate the reactive forces. To determine these conditions, the body is isolated and all the active and reactive forces acting on it are determined. When the isolated body is drawn with all the forces and reactive forces acting on it, such a diagram is known as a free body diagram. Note that in a complex structural system, the free body may be drawn for the whole system or a part of it. The free body diagram provides an excellent way of book-keeping in a structural problem by ensuring that all the forces and reactive forces are taken into account. The following problems illustrate the use of free body diagrams.

Example 1.1 Free body diagram

The drum shown in Fig. 1.22(a) weighs 800 N and is being rolled over a step 4 cm high. Determine the value of *P* required to roll it over the step if it is applied horizontally at the centre of the drum. Also determine the minimum value of *P* required to pull the drum if it can be inclined at any angle α.

Solution The free body diagram of the drum is shown in Fig. 1.22(b). Assuming the surfaces to be smooth, there is a reaction *R* at the floor, a reaction *Q* at the point of contact with the step, and the pull *P* in addition to the weight of the drum acting through its CG. Note that when the drum is about to be pulled over the step, it leaves contact with the ground and *R* becomes zero. There are then only three forces acting on the body, 800 *N*, *P*, and *Q*. The direction of *Q* will be such that it passes through the centre of the drum. In Fig. 1.22(b),

$$\sin\theta = \frac{16}{20} = 0.8, \quad \cos\theta = \frac{12}{20} = 0.6$$

$$\Sigma Fx = 0, \ \overset{\rightarrow}{+}, \ P - Q\cos\theta = 0$$

$$\Sigma Fy = 0, \ \uparrow+, \ Q\sin\theta - 800 = 0, \ Q = 1000 \text{ N}, \quad P = 600 \text{ N}$$

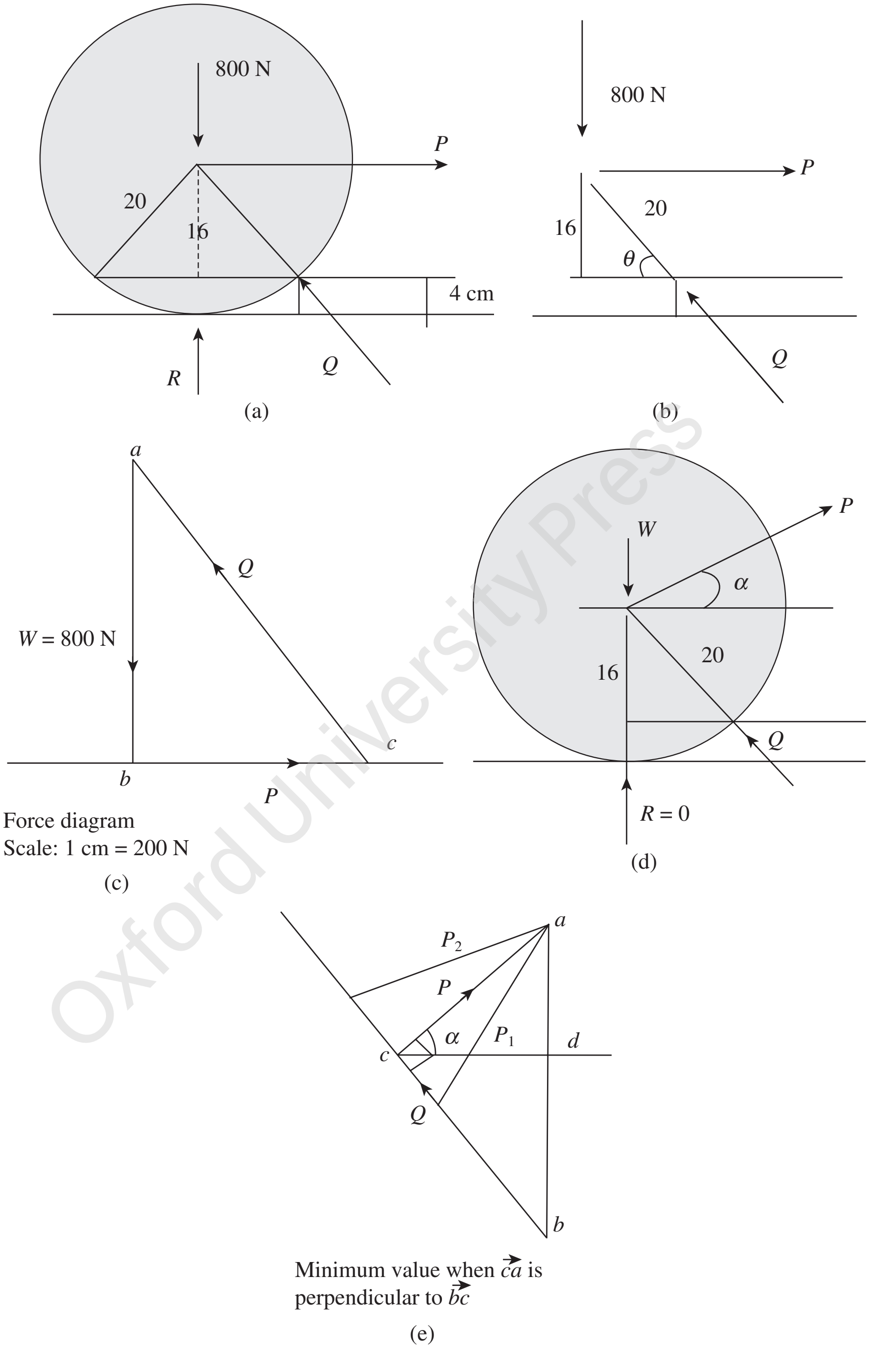


Fig. 1.22

Graphically, since the body is acted upon by a three-force system, the solution is found by drawing a force triangle shown in Fig. 1.22(c). $\overrightarrow{ab} = W = 800$ N to scale. Draw a line parallel to P through b and a line parallel to Q through a, intersecting at C. $\overrightarrow{bc} = P$ and $\overrightarrow{ca} = Q$ can be measured to scale.

To determine the minimum value of P inclined at an angle a, from Fig. 1.22(d), $P \cos\alpha = 0.6\,Q$; $W - P \sin\alpha = 0.8\,Q$. To eliminate Q, $4P \cos\alpha = 2.4\,Q$; $3W - 3P \sin\alpha = 2.4\,Q$. Subtracting,

$$4P\cos\alpha - 3W + 3P\sin\alpha = 0$$
$$P(4\cos\alpha + 3\sin\alpha) = 3W$$

For P to be minimum, $(4\cos\alpha + 3\sin\alpha)$ has to be maximum. Therefore,

$$\frac{d}{d\alpha}(4\cos\alpha + 3\sin\alpha) = 0$$
$$-4\sin\alpha + 3\cos\alpha = 0$$
$$\tan\alpha = 0.75, \quad \alpha = 36.87^\circ$$
$$P = \frac{3 \times 800}{4\cos 36.87 + 3\sin 36.87} = 480 \text{ N is the minimum value of } P.$$

Graphically, as in Fig. 1.22(e), we lay out $\overrightarrow{ab}$ = 800 N to scale. Draw a line parallel to the line of action of Q from b. Any line drawn from a to intersect this line gives a value of P and its corresponding direction. The minimum value of P will be obtained when ac is at right angles to bc. Therefore, draw a perpendicular to bc from a. $\overrightarrow{ca}$ gives the minimum value of P and α can be measured from this triangle as $\overrightarrow{ca}$ and angle ad.

Example 1.2 Free body diagram

Two pipes 20 cm ϕ, weighing 1200 N and 16 cm ϕ, weighing 800 N, lie in a trench as shown in Fig. 1.23(a). Draw a free body diagram for the two pipes together and for each individual pipe. Assume the contacting surfaces as rough.

Solution Figures 1.23(b), (c), and (d) show the free body diagram. Note that when the free body diagram of the two pipes together is drawn, the forces Q vanish.

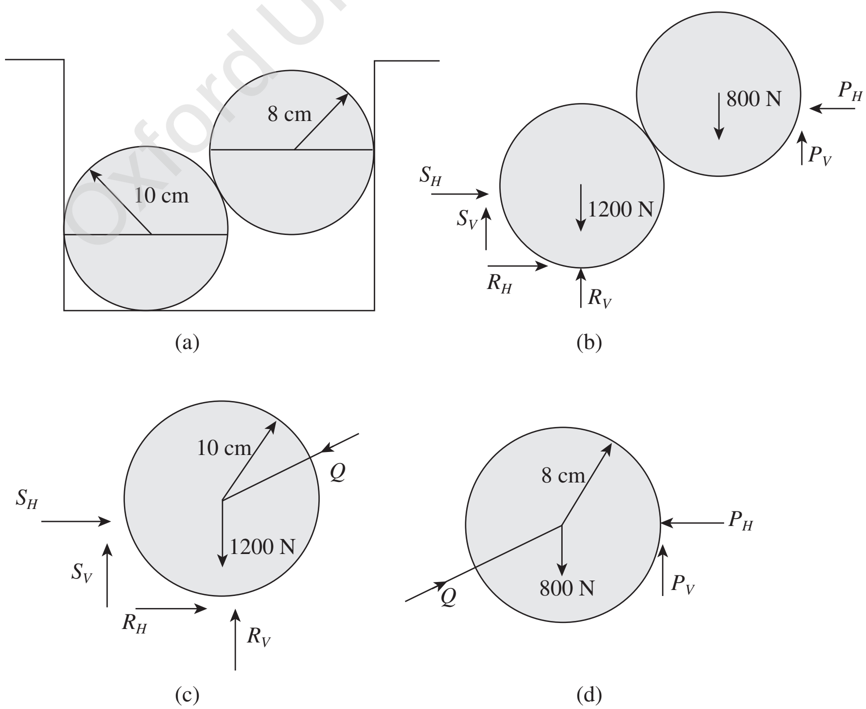


Fig. 1.23

Example 1.3 Reactive forces in a beam

The rod *PQ* shown in Fig. 1.24 is acted upon by two forces, one of 600 N and another of 900 N. It has a hinged support at *P* and a roller support at *Q*. Find the reactive forces at the supports.

Solution Selecting the *X*- and *Y*-directions as shown, from the free body diagram illustrated in Fig. 1.24(b),

$$\Sigma Fx = 0,\ \overset{\rightarrow}{+},\ R_{PH} = 0$$

$$\Sigma Fy = 0,\ \uparrow +,\ R_{PV} - 600 - 900 + R_{QV} = 0$$

$$\Sigma M = 0 \text{ about } P +,\quad 600 \times 2 + 900 \times 5 - R_{QV}\, 6 = 0$$

From the last equation, $R_{QV} = 950$ N.

$$R_{PV} = 1500 - 950 = 550 \text{ N}$$

Graphical Solution The graphical solution is shown in Fig. 1.24(c). We lay out the rod and the positions of the forces to a linear scale, and then draw the force polygon *a-b-c* to a load scale. Select a

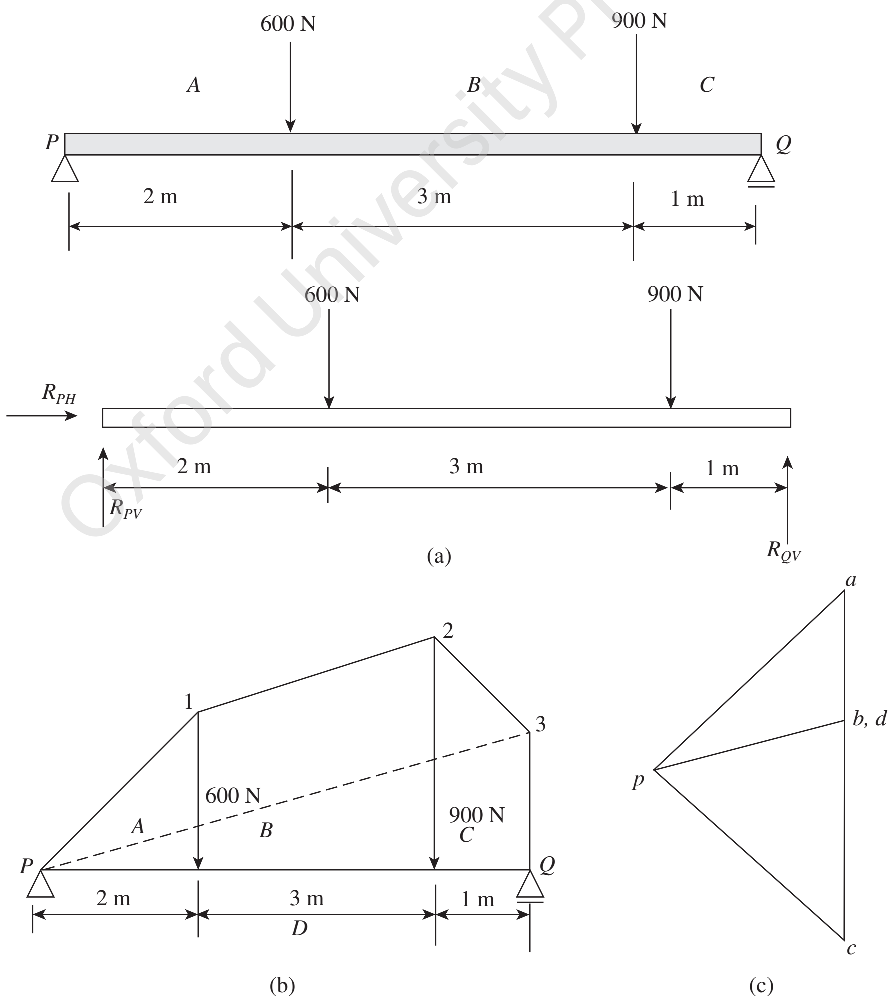


Fig. 1.24

pole p and draw pa, pb, and pc. Then, draw lines parallel to these in the space diagram. Consider R_{PH} and R_{PV} as a combined force R_A whose direction is unknown but which passes through P. Start the funicular polygon from P to get P-1-2-3. The last segment 2-3 ends where it intersects reaction R_{QV} (vertical line through Q). Draw P-3. Draw a line parallel to P-3 through p which intersects the vertical line through c at d. $\overrightarrow{cd} = R_{QV}$ and $\overrightarrow{da} = R_P$. Since R_P is vertical, $R_{PH} = 0$.

Example 1.4 Reactive forces in a frame

Determine the reactions at the supports of the structure carrying load as shown in Fig. 1.25(a).

Solution The free body diagram for the structure is shown in Fig. 1.25(b). The reactive components at A are R_{AH} and R_{AV}, and the reactive component at D is R_D, perpendicular to the plane of the roller and inclined at 60° to the horizontal.

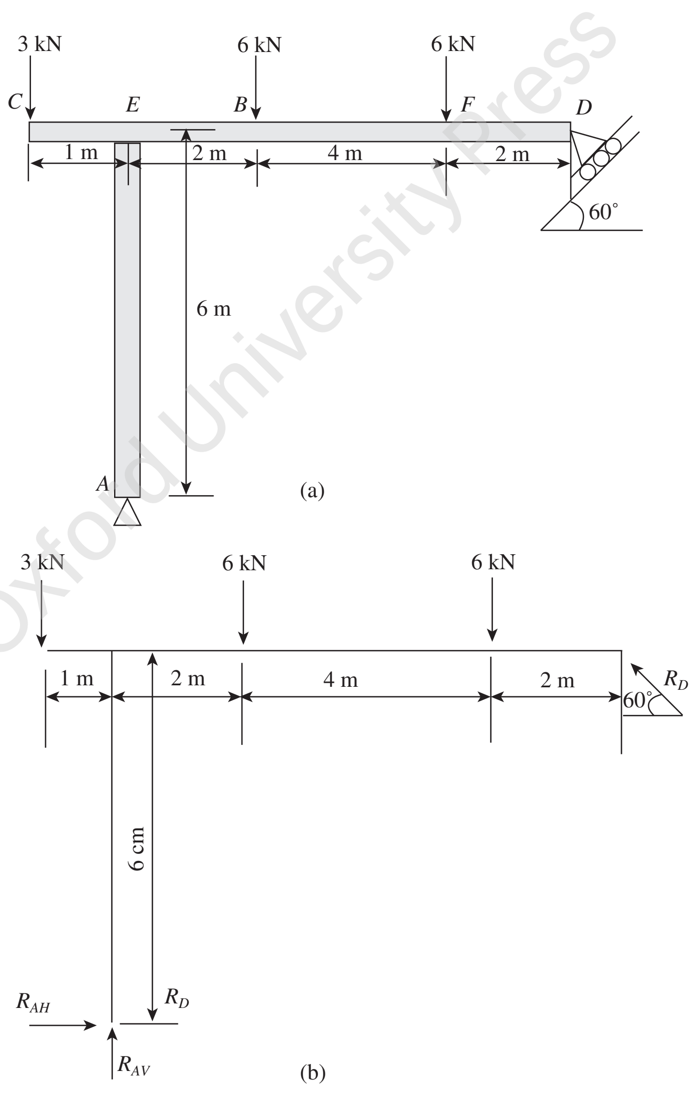


Fig. 1.25

Applying the conditions of equilibrium,

$\Sigma Fx = 0, \overset{\rightarrow}{+}, \; R_{AH} - R_D \cos 60° = 0$

$\Sigma Fy = 0, \uparrow +, \; R_{AV} + R_D \sin 60° - 3 - 6 - 6 = 0$

$\Sigma M = 0 \text{ at } A, \; -3 \times 1 + 6 \times 2 + 6 \times 6 - R_D \cos 60° \times 6 - R_D \sin 60° \times 8 = 0$

From the last equation, $R_D = 4.53$ kN, $R_{AH} = 2.27$ kN, and $R_{AV} = 11.07$ kN.

1.6 LOADS ON STRUCTURES

Structures are subjected to different types of loads and due to different causes. Loads on structures can be classified into (i) dead loads, which include the self-weight of the structure and other fixed loads; (ii) live loads, which do not have a fixed position and can be placed anywhere for maximum effect; (iii) wind loads; (iv) snow loads; (v) seismic loads to take into account the loads due to earthquakes; (vi) impact loads such as moving vehicles, crane loads, machines, etc.

Depending upon their distribution, loads may be classified in different ways. Let us discuss these briefly.

Point load or concentrated load Any load acts on a finite area and not at a point. However, if a large load acts on a small area, it is considered as a point load [Fig. 1.26(a)].

Distributed loads The different types of distributed loads are shown in Fig. 1.26(b). Such loads may be uniformly distributed, uniformly varying, and non-uniformly varying. The variation is in the intensity of the load at a point and can be mathematically expressed as shown in the figure. The total load acts through the CG of the distribution figure. In these figures, l is the length of the load, w_x is the intensity (load per unit length) at a distance x, and w is the load intensity (known) at a distance l.

Couple load A couple load is represented as shown in Fig. 1.26(c), giving the location and magnitude of the couple M. The couple has the same moment about any point in its plane and hence should be included in all moment equations. The couple does not appear in force equations as the resultant force in a couple is zero.

Combination of loads Many structural elements in practice will be subjected to a variety of loads. Such structural elements are analysed for loads like self-weight, point loads, varying loads, etc. A typical structural element is shown in Fig. 1.26(d) acted upon by a combination of loads.

1.7 CENTROID

The centroid, as you would have learnt in a first course in mechanics, is a point in the area where the whole area can be assumed to be concentrated. The concept of the centroid is analogous to that of the centre of gravity in the case of a mass. The centre of gravity is a point in a body where the whole mass of the body can be assumed to be concentrated. Mathematically,

$$\bar{x} = \frac{\int (\delta M)\, x}{\int \delta M}$$

$$\bar{y} = z \text{ and } \quad \bar{z} = \frac{\int (\delta M) z}{\int \delta M} \quad \text{[see Fig. 1.27(a)]}$$

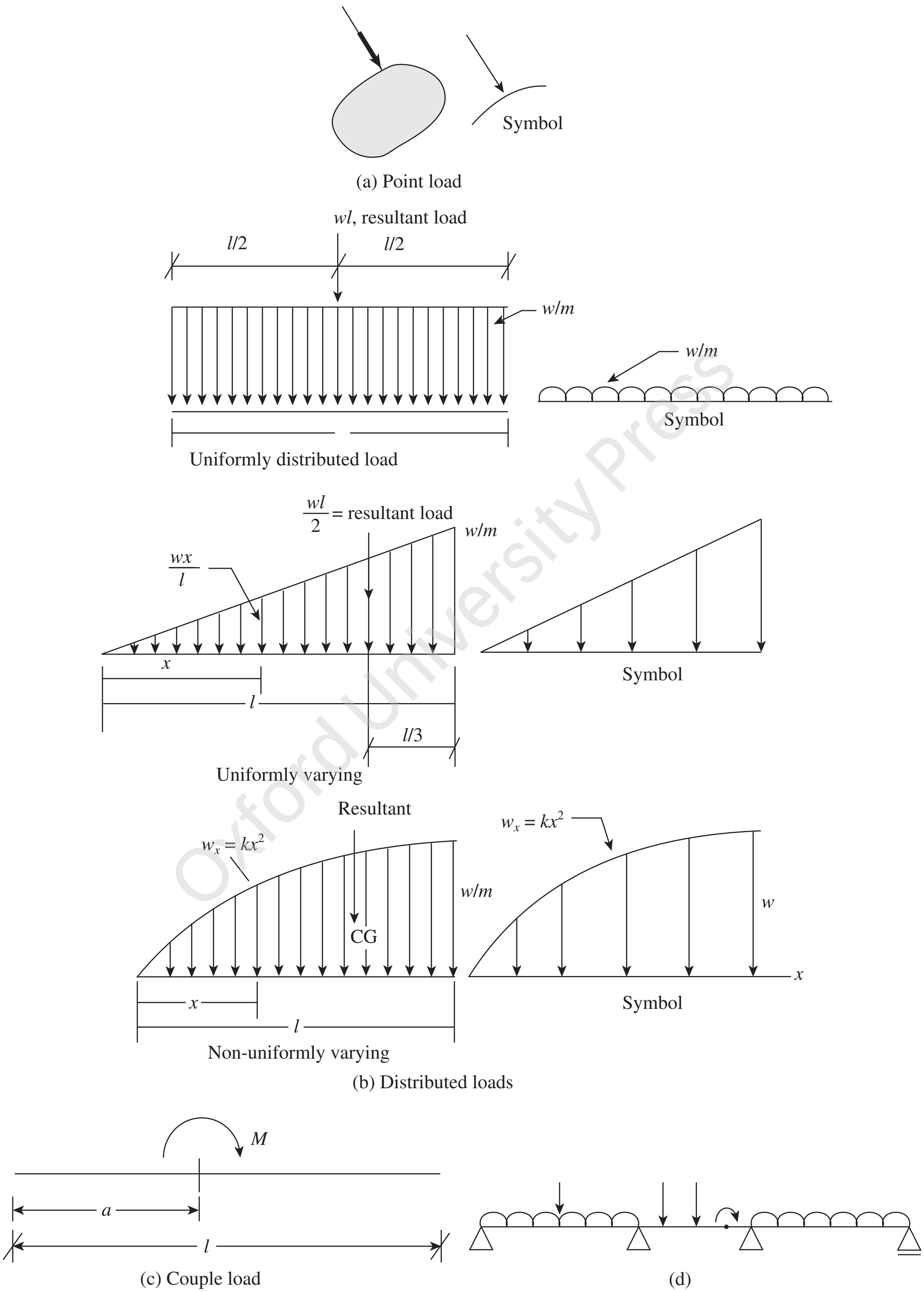

Symbol
(a) Point load
wl, resultant load
l/2
l/2
w/m
w/m
Symbol
Uniformly distributed load
wl/2 = resultant load
w/m
wx/l
x
l
l/3
Symbol
Uniformly varying
Resultant
$w_x = kx^2$
$w_x = kx^2$
w/m
CG
w
x
Symbol
x
l
Non-uniformly varying
(b) Distributed loads
M
a
l
(c) Couple load
(d)

Fig. 1.26

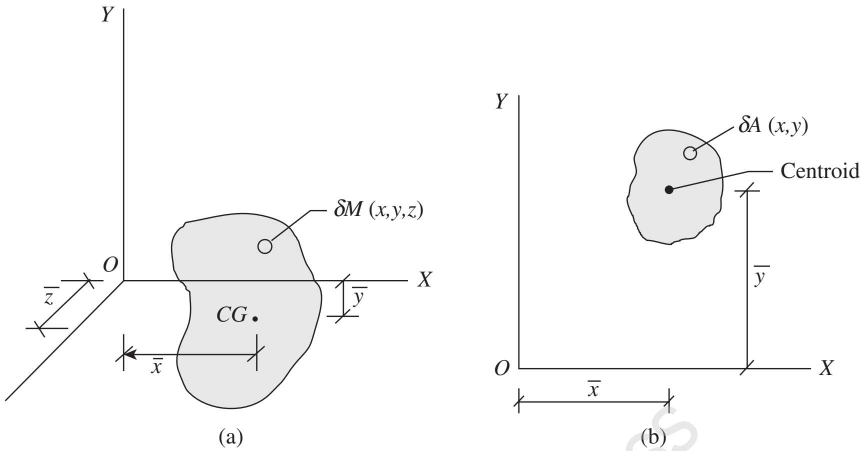


Fig. 1.27

Similarly,

$$\bar{x} = \frac{\int (\delta A)x}{\int \delta A} \quad \text{and} \quad \bar{y} = \frac{\int (\delta A)y}{\int \delta A}$$

for an area A [Fig. 1.27(b)].

Here $\bar{x}$, $\bar{y}$, and $\bar{z}$ are distances to the centre of gravity (CG) of the mass from a set of X-Y-Z coordinate axes and x, y, z are the coordinates of an elementary mass. Similarly, $\bar{x}$ and $\bar{y}$ are distances to the centroid of the area from the X-Y coordinate axes and (x, y) are the coordinates of an elementary area.

The problem of computing the centroid of an area is frequently encountered in strength of materials. Table 1.1 in Appendix 1 gives the centroids of common geometric areas. These are useful in locating the centroids of composite areas which are made up of simple geometric shapes. In the case of complex shapes, the centroids can be located by integration as given above. The following two examples illustrate the procedure.

Example 1.5 Centroid of channel section

Find the centroid of the channel section shown in Fig. 1.28.

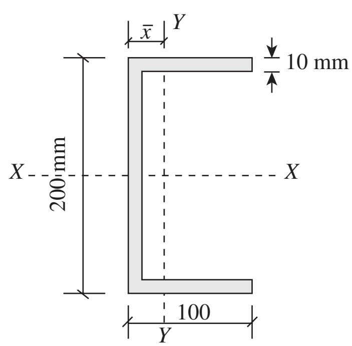


Fig. 1.28

Solution The channel section is symmetrical about a horizontal axis through the middle of the depth, i.e. the X-X axis shown in Fig. 1.28. Therefore, $\bar{y}$ = 100 mm, we have to find $\bar{x}$.

From Fig. 1.28,

$$\bar{x} = \frac{2 \times 100 \times 10 \times 50 + 180 \times 10 \times 5}{2 \times 100 \times 10 + 180 \times 10}$$

$$= 28.68 \text{ mm}$$

Example 1.6 Centroid of semicircle and quarter circle areas

Find the centroid of a semicircular area and the quarter circle shown in Figs. 1.29(a) and (b), respectively.

Solution Considering an elementary area subtending an angle $\delta\theta$ at the centre, the area of the hatched portion,

$$\text{Elementary area} = Rd\theta\,\frac{R}{2} = \frac{R^2 d\theta}{2}$$

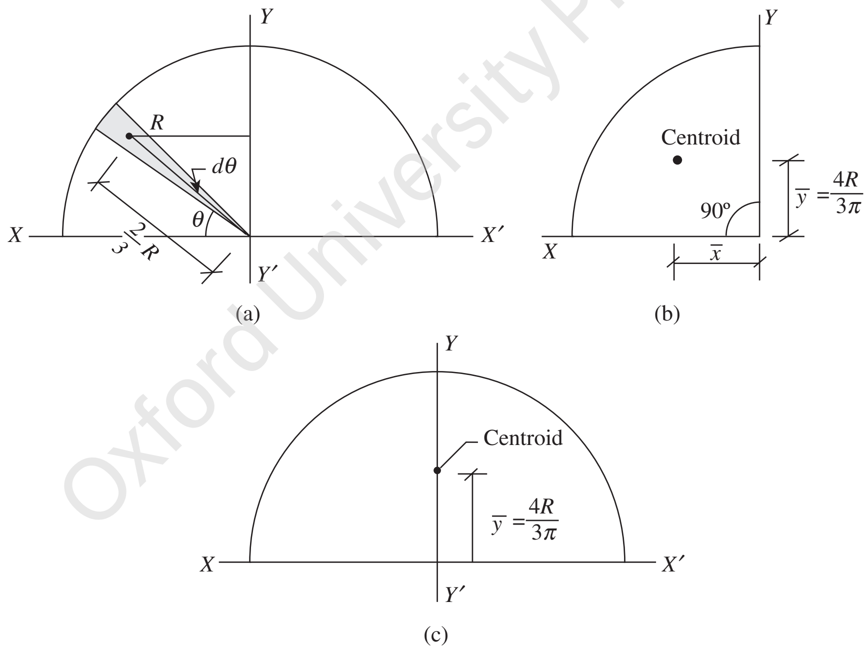


Fig. 1.29

This can be assumed to act at $2R/3$ from the centre O. The semicircle is symmetrical about a vertical diameter. We have to find $\bar{y}$ (the distance from the X-axis).

$$\text{Moment of elementary area} = \frac{R^2 d\theta}{2}\,\frac{2}{3}\,R\sin\theta$$

$$= \frac{R^3}{3}\sin\theta\, d\theta$$

For a semicircle, this expression can be integrated from $\theta = 0$ to $\theta = \pi$, and for the quarter circle from $\theta = 0$ to $\theta = \pi/2$. This will give the moment of the total area, which is equal to (area × $\bar{y}$) for the areas.

Semicircle

$$\frac{\pi R^2}{2}\bar{y} = \int_0^{\pi} \frac{R^3}{3}\sin\theta\, d\theta = \frac{R^3}{3}[-\cos\theta]_0^{\pi}$$

$$= \frac{R^3}{3}(1+1) = \frac{2R^3}{3}$$

$$\bar{y} = \frac{2R^3 \times 2}{3\pi R^2} = \frac{4R}{3\pi}$$

Quarter circle

$$\frac{\pi R^2}{4}\bar{y} = \int_0^{\pi/2} \frac{R^3}{3}\sin\theta\, d\theta = \frac{R^3}{3}[-\cos\theta]_0^{\pi/2}$$

$$= \frac{R^3}{3}[-0+1] = \frac{R^3}{3}$$

$$\bar{y} = \frac{4R}{3\pi}$$

In the case of a quarter circle, from considerations of symmetry, $\bar{x} = \bar{y}$. The location of the centroid of a semicircle is shown in Fig. 1.29(c), and that of a quarter circle in Fig. 1.29(b).

1.8 MOMENT OF INERTIA

The reader would have already come across an expression (in physics or mechanics) $\int dMy^2$, particularly in connection with the rotation of solids about an axis. In Fig. 1.30, dM is an elementary mass and y is the distance of this elementary mass from an axis. This expression when evaluated gives a quantity which is called moment of inertia. It is usually denoted by the symbol I. Subscripts are used to indicate the axes about which the moments are taken. For example, I_{XX} represents the moment of inertia about the X-X axis.

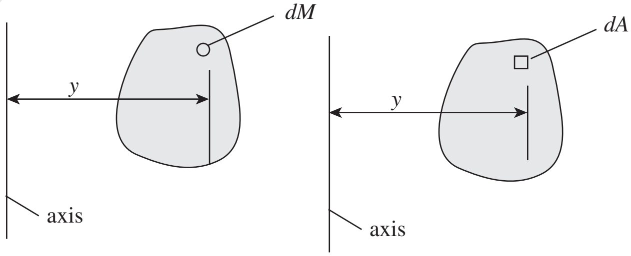


Fig. 1.30

A similar expression occurs in many structural calculations as $\int dA\ y^2$, where dA is an elementary area and y is the distance of dA from a given axis. The term moment of inertia does not apply to areas. The term used for this quantity is *second moment of area*. However, through long usage, these two terms are used to denote second moment of area.

This is a very important quantity for structural engineers, finding application in the design of beams, columns, etc

1.9 COMPUTATION OF SECOND MOMENT OF AREA

The second moment of area is calculated from the integral $\int dA\, y^2$. The summation can be done by integration in the case of geometrical shapes like rectangle, triangle, circle, etc. Since the area is in L^2 units and y is in L units, the unit of moment of inertia will be in L^4 units, i.e., mm^4, m^4, etc. The following examples illustrate the basic procedure

Example 1.7 Moment of inertia of a rectangle

Find the moment of inertia of a rectangle of width b and depth d about a horizontal axis through its centroid and about a parallel axis through its base (Fig. 1.31)

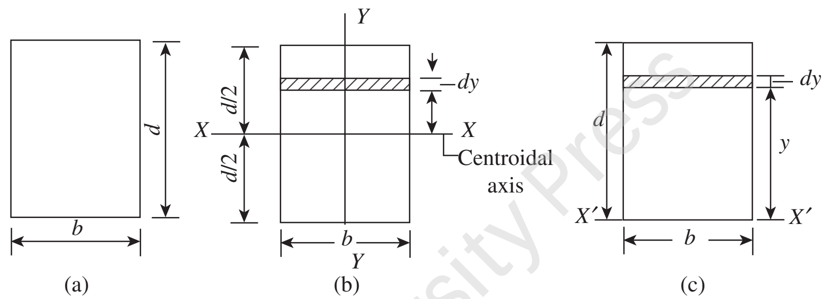


Fig. 1.31

Solution As shown in Fig. 1.31(b), consider an elementary strip of thickness dy at a distance y from the centroid axis X-X.

Area of the strip = $b\,dy$

MI about axis X-X = $b\,dy\,y^2$

y has a range from $-d/2$ to $+d/2$. Therefore,

$$I_{XX} = \int_{-d/2}^{+d/2} b\,dy\,y^2 = b\left[\frac{y^3}{3}\right]_{-d/2}^{+d/2} = \frac{bd^3}{12}$$

Similarly, it can be shown that

$$I_{YY} = \frac{db^3}{12}$$

To determine the MI about a parallel axis through the base, from Fig. 1.31(c),

$$I_{X'-X'} = \int_0^b bdy\,y^2 = b\left[\frac{y^3}{3}\right]_0^d = \frac{bd^3}{3}$$

1.9.1 Parallel Axis Theorem

In Example 1.7, we found the MI about a parallel axis from the fundamental principles. A general theorem to find MI about parallel axes is stated as follows.

If I_{GG} is the MI of an area about an axis through its centroid, then the MI about any axis A-A at a distance d from GG and parallel to it is $I_{AA} = I_{GG} + A\,d^2$, where A is the area of the figure.

Considering Fig. 1.32, the MI of the elementary strip about the axis A-A is $dA(y + d)^2$. The MI of the whole area about the axis A-A is

$$I_{AA} = \int dA(y + d)^2 = \int dA(y^2 + d^2 + 2yd)$$

$$= \int dA\, y^2 + \int dA\, d^2 + \int dA \times 2yd$$

$$= I_{GG} + d^2 \int dA + 2d \int dA\, y$$

$$\int dAy = 0$$

being the moment of the area about its centroid

$$\int dA = A$$

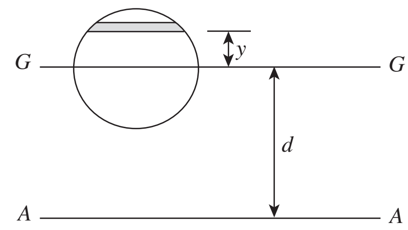


Fig. 1.32

Therefore,

$$I_{AA} = I_{GG} + Ad^2$$

This is known as the parallel axis theorem and is very useful in finding the MIs of composite areas.

Example 1.8 Moment of inertia of a triangle

Find the MI of a triangular area about (i) an axis through its base and (ii) an axis through the CG and parallel to the base.

Solution The triangle is shown in Fig. 1.33(a). Considering a strip of width w at a distance of y from vertex A, $w = by/h$, area = $(by/h)dy$, and $I_{XX} = (by/h)dy(h - y)^2$ for the elementary strip. For the whole area,

$$I_{XX} = \frac{b}{h}\int_0^b ydy\,(h-y)^2 = \frac{b}{h}\int_0^h y(h^2 + y^2 - 2hy)dy$$

$$= \frac{b}{h}\left[\frac{h^2y^2}{2} + \frac{y^4}{4} - 2h\frac{y^3}{3}\right]_0^h = \frac{bh^3}{12}$$

Considering Fig. 1.33(b), for the MI about axis G-G, at $h/3$ from the base,

$$I_{GG} = \int_0^h \frac{by}{h}dy\left(\frac{2}{3}h - y\right)^2 = \frac{b}{h}\int_0^h y\left(\frac{4}{9}h^2 + y^2 - \frac{4}{3}hy\right)dy$$

$$= \frac{b}{h}\left(\frac{4}{9}\frac{h^2y^2}{2} + \frac{y^4}{4} - \frac{4}{3}h\frac{y^3}{3}\right)_0^h = \frac{bh^3}{36}$$

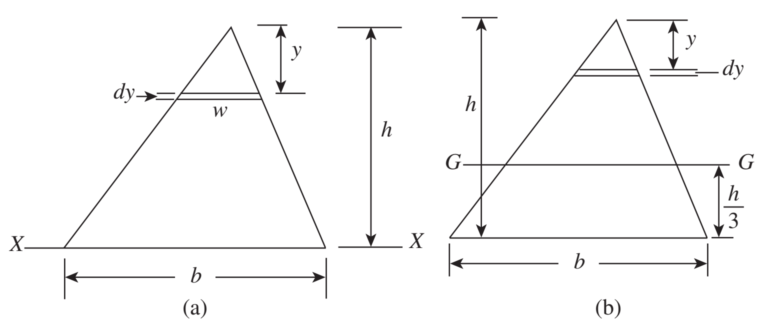


Fig. 1.33

Knowing I_{XX}, I_{GG} could have been found by the parallel axis theorem as

$$I_{GG} = I_{XX} - Ad^2 = \frac{bh^3}{12} - \frac{bh}{2}\left(\frac{1}{3}h\right)^2 = \frac{bh^3}{12} - \frac{bh^3}{18} = \frac{bh^3}{36}$$

1.9.2 Perpendicular Axes Theorem

We have seen how to determine the second moment of area about an axis lying in the plane of the area. In some structural calculations, we need to find the second moment of an area about an axis passing through the centroid and perpendicular to the plane containing the area. This is required in the case of torsion of members (Chapter 7). This is calculated using the perpendicular axes theorem.

The perpendicular axes theorem states that "*the second moment of an area about an axis perpendicular to the plane of the area through a point is equal to the sum of the second moment of areas about two mutually perpendicular axes through that point.*"

The second moment of area about an axis perpendicular to the plane of the area is known as polar moment of inertia.

1.9.3 Polar Moment of Inertia

The polar moment of inertia is defined as the MI about an axis perpendicular to the plane of the area. Polar MI is denoted by the symbol J.

Considering Fig. 1.34, $J_{ZZ} = \int_A dA\, r^2$. If X and Y are two axes in the plane of the area, mutually perpendicular and passing through O, then

$$J_{ZZ} = \int_A dA\, r^2 = \int_A dA(x^2 + y^2) = \int_A dA\, x^2 + \int dA\, y^2$$
$$= I_{XX} + I_{YY}$$

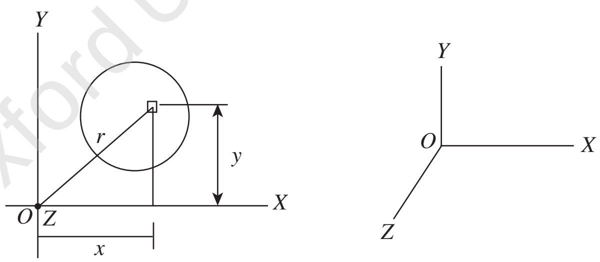


Fig. 1.34

Example 1.9 Moment of inertia of a circle

Determine the MI of a circular area about one of its diameters.

Solution Referring to Fig. 1.35, for the elementary strip parallel to the diametrical axis G-G, width $b = 2r\cos\theta$, $y = r\sin\theta$, $dy = r\cos\theta\, d\theta$,

$$I_{GG} \text{ of elementary strip} = dA\, y^2 = b\, dy\, y^2$$
$$= 2r\cos\theta\, r\cos\theta\, d\theta\, r^2\sin^2\theta$$

As y varies from $-r$ to $+r$, θ varies from $-\pi/2$ to $+\pi/2$. For the whole area,

$$I_{GG} = \int_{-\pi/2}^{\pi/2} 2r^4 \sin^2\theta\cos^2\theta\, d\theta$$
$$= 4r^4 \int_0^{\pi/2} \sin^2\theta\,(1 - \sin^2\theta)\, d\theta$$

$$= 4r^4 \left[\int_0^{\pi/2} \sin^2\theta \, d\theta - \int_0^{\pi/2} \sin^4\theta \, d\theta\right]$$

$$= \frac{\pi r^4}{4}$$

Since $r = d/2$,

$$I_{GG} = \frac{\pi (d/2)^4}{4} = \frac{\pi d^4}{64}$$

This result can be easily obtained from $J_{ZZ} = I_{XX} + I_{YY}$. In Fig. 1.35(b), taking X- and Y-axes as shown, $I_{XX} = I_{YY}$ from considerations of symmetry for the elementary strip in the form of the ring shown, at a distance y from O.

$$J_{ZZ} = 2\pi y dy \, y^2$$

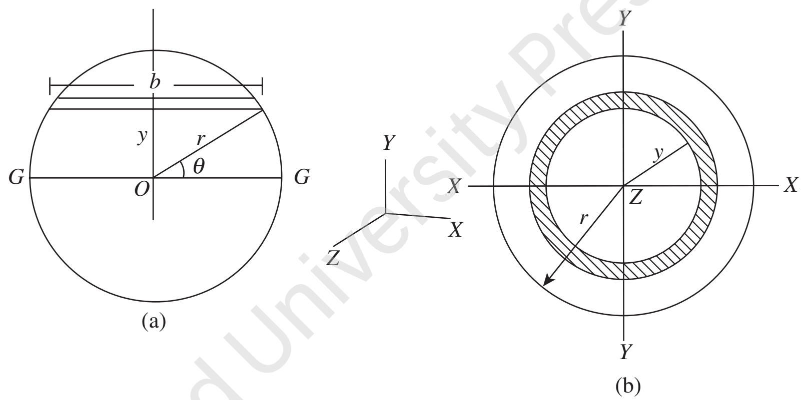


Fig. 1.35

For the whole area,

$$J_{ZZ} = \int_0^r 2\pi y^3 dy = \frac{\pi r^4}{2}$$

$$I_{XX} = I_{YY} = \frac{1}{2}\frac{\pi r^4}{2} = \frac{\pi r^4}{4}$$

Example 1.10 MI of a parabolic area

For the parabolic area shown in Fig. 1.36, find the MI with respect to the X- and Y-axes.

Solution In Fig. 1.36(b),

Width of elementary strip $= a - x = a - ky^2$

Area $= (a - ky^2)dy$, $\quad I_{XX} = (a - ky^2)\, dy\, y^2$

Integrating over the whole area,

$$I_{XX} = \int_0^b (a - ky^2)\, y^2\, dy = \left(\frac{ay^3}{3} - \frac{ky^5}{5}\right)_0^b = \frac{ab^3}{3} - \frac{kb^5}{5}$$

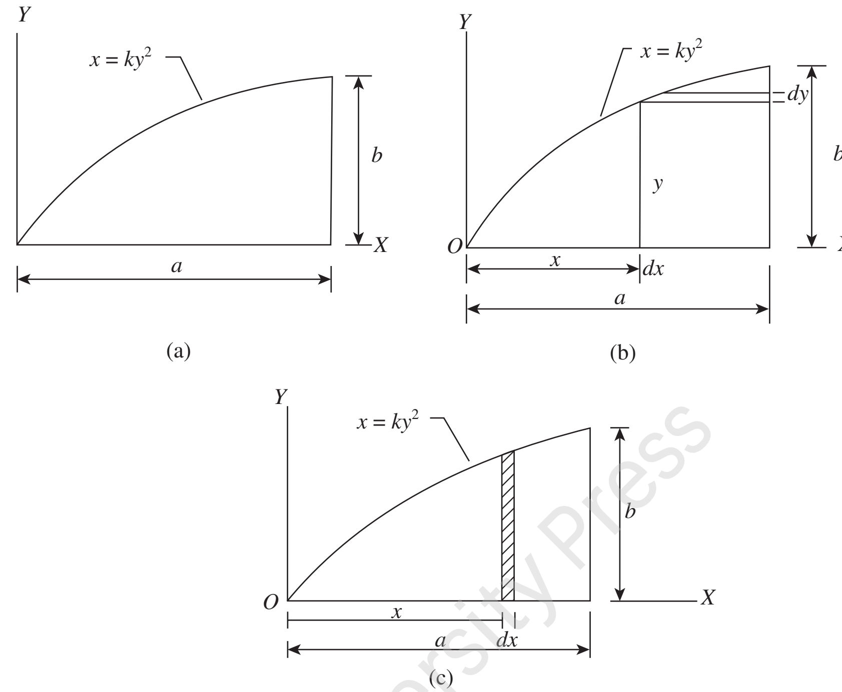


Fig. 1.36

Since $a = kb^2$, $b^2 = a/k$.

$$I_{XX} = \frac{ab^3}{3} - \frac{ab^3}{5} = \frac{2}{15}ab^3$$

To calculate I_{YY}, consider an elementary strip parallel to the Y-axis. For the elementary strip, area $= ydx = y \times 2ky\ dy$, $\quad I_{YY} = 2ky^2dy\ x^2$. Therefore,

$$I_{YY} = 2ky^2dyk^2\ y^4$$

Integrating over the whole area,

$$I_{YY} = \int_a^b 2k^3 ybdy = 2k^3\left(\frac{y^7}{7}\right)_0^b = \frac{2k^3b^7}{7} = \frac{2}{7}ba^3$$

1.9.4 Moment of Inertia of a Composite Area

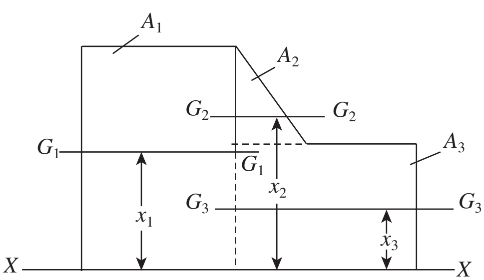


Fig. 1.37

The second moment of area composed of a number of simple areas can be found as the sum of the second moments of area of its parts. As the second moment of an area will never be negative, simple summation can be done. For example, considering the area shown in Fig. 1.37, the second moment of area about the XX-axes can be calculated as

$$I_{XX} = I_{G_1G_1} + A_1x^2{}_1 + I_{G_2G_2} + A_2\,x^2{}_2 + I_{G_3G_3} + A_3\,x^2{}_3$$

The following examples illustrate the procedure.

Example 1.11 MI of a T-section

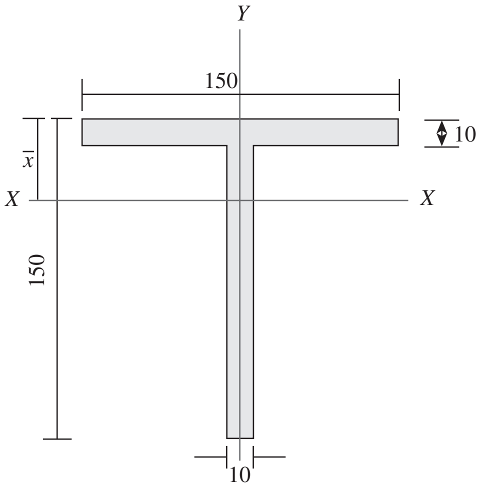


Fig. 1.38

Find the MI through the centroidal axes *X-X* and *Y-Y* for the T-shaped section shown in Fig. 1.38. (All lengths are in mm.)

Solution The section is symmetrical about the *y*-axis. To locate the distance $\bar{x}$ (from the upper edge), take the moment about the top edge of the T-section of the areas:

$$(150 \times 10 + 140 \times 10)\,\bar{x} = 150 \times 10 \times 5 + 140 \times 10 \times 80$$

$$\bar{x} = 41.2 \text{ mm}$$

$$I_{XX} = 150 \times \frac{10^3}{12} + 150 \times 10\,(41.2 - 5)^2$$

$$+ \frac{10 \times 140^3}{12} + 140 \times 10\,(108.8 - 70)^2$$

$$= 6.372 \times 10^6 \text{ mm}^4$$

$$I_{YY} = \frac{10 \times 150^3}{12} + \frac{140 \times 10^3}{12} = 2.8242 \times 10^6 \text{ mm}^4$$

Example 1.12 MI of an unequal angle section

Find the MI about the horizontal and vertical axes passing through the centroid of the unequal angle section shown in Fig. 1.39.

Solution The angle section is unsymmetrical in both directions. Here $\bar{x}$ as well as $\bar{y}$ have to be determined. Taking moments about the left edge of the areas,

$$(100 \times 10 + 110 \times 10)\,\bar{x} = 100 \times 10 \times 5 + 110 \times 10 \times 65$$

$$\bar{x} = 36.43 \text{ mm}$$

Similarly, taking moments about the bottom edge,

$$(100 \times 10 + 110 \times 10)\,\bar{y} = 100 \times 10 \times 50 + 110 \times 10 \times 5$$

$$\bar{y} = 26.43 \text{ mm}$$

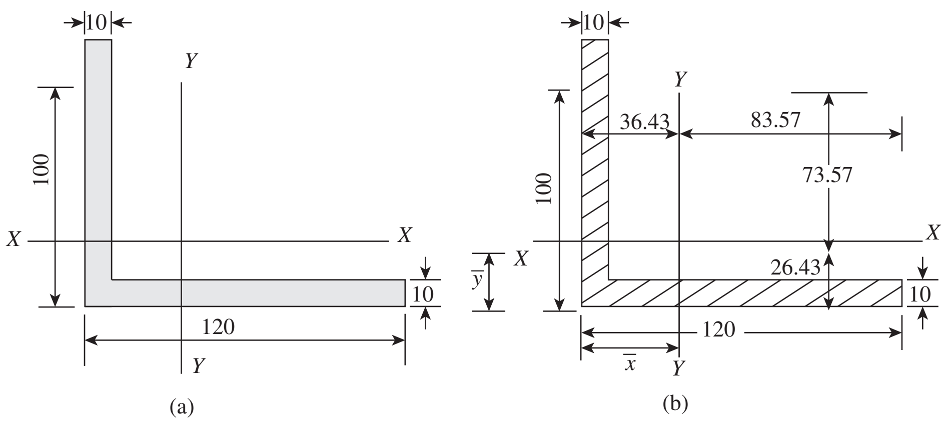


Fig. 1.39

From the distances shown in Fig. 1.39(b),

$$I_{XX} = \frac{10 \times 100^3}{12} + 10 \times 100\,(73.57 - 50)^2 + \frac{110 \times 10^3}{12} + 110 \times 10\,(26.43 - 5)^2$$

$$= 1.9032 \times 10^6 \text{ mm}^4$$

$$I_{YY} = \frac{10 \times 100^3}{12} + 100 \times 10\,(36.43 - 5)^2 + \frac{10 \times 100^3}{12} + 10 \times 110\,(83.57 - 55)^2$$

$$= 3.0032 \times 10^6 \text{ mm}^4$$

Example 1.13 MI of a double channel section

Find the MI of the channel section shown in Fig. 1.40(a) about the *X-X* and *Y-Y* axes passing through its centroid. If two such sections are kept back to back as shown in Fig. 1.40(b), find the distance *d* such that $I_{XX} = I_{YY}$ for the compound section.

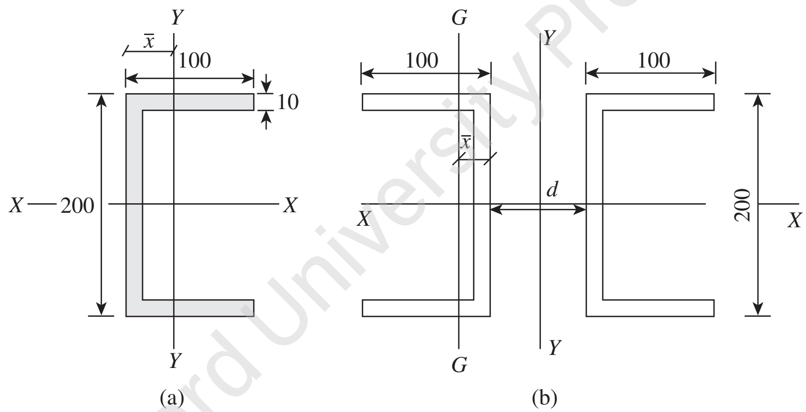


Fig. 1.40

Solution The channel section is symmetrical about a horizontal axis through its mid-height. To locate the *Y-Y* axis, we have to calculate $\bar{x}$. Taking moments about the left vertical edge,

$$(200 \times 10 + 90 \times 10 + 90 \times 10)\,\bar{x} = 200 \times 10 \times 5 + 2 \times 90 \times 10 \times 55$$

$$\bar{x} = 28.68 \text{ mm}$$

$$I_{XX} = \frac{10 \times 200^3}{12} + 2\left[\frac{90 \times 10^3}{12} + 90 \times 10 \times 95^2\right] = 22.93 \times 10^6 \text{ mm}^4$$

$$I_{YY} = \frac{200 \times 10^3}{12} + 200 \times 10 \times (28.68 - 5)^2 + 2\left[\frac{10 \times 90^3}{12} + 10 \times 90 \times 26.32^2\right] = 3.6 \times 10^6 \text{ mm}^4$$

In Fig. 1.40(b), I_{XX} remains the same for each channel. So, $I_{XX} = 2 \times 22.93 \times 10^6 = 45.86 \times 10^6$ mm^4 for the whole section.

$$I_{YY} = 2 \times \left[I_{GG} \text{ for each channel} + \text{area of channel } \times \left(\bar{x} + \frac{d}{2}\right)^2\right]$$

$$= 2\left[3.6 \times 10^6 + 3800 \times \left(28.68 + \frac{d}{2}\right)^2\right] = 7.2 \times 10^6 + 7600\left(28.68^2 + \frac{d^2}{4} + 28.68d\right)$$

This is equal to I_{XX}.

$$7.2 \times 10^6 + 6.2513 \times 10^6 + 7600\ \frac{d^2}{4} + 217{,}968d = 45.86 \times 10^6$$

$$1900d^2 + 217{,}968d - 32.41 \times 10^6 = 0$$

$$d = 85.3 \text{ mm}$$

Example 1.14 MI of a rectangular lamina with hole

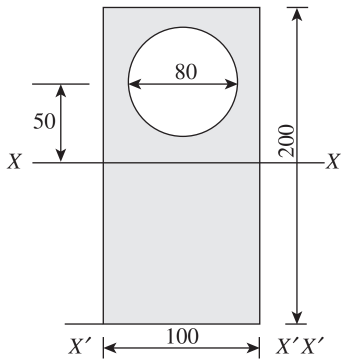


Fig. 1.41

From a rectangular plate, 100 mm × 200 mm, a circular portion is removed as shown in Fig. 1.41. Find the MI of the plate about an axis through its base.

Solution The circular portion can be taken as a negative area and its MI subtracted from that of the full rectangular plate.

$$I_{X'X'} = 100 \times \frac{200^3}{3} - \frac{\pi \times 80^4}{64} - \frac{\pi \times 80^2}{4} \times 150^2$$

$$= 1.5156 \times 10^8 \text{ mm}^4$$

Example 1.15 MI of a composite area

Find the MI of the composite area shown in Fig. 1.42(a) about the X-X and Y-Y axes.

Solution The distances of the centroids of each part of the area from the respective reference lines are shown in Fig. 1.42(b). I_{XX} and I_{YY} can be calculated as

$$I_{XX} = \frac{200 \times 100^3}{3} + \frac{\pi \times 50^4}{8} + \frac{\pi \times 50^2}{2} \times 50^2 + \frac{200 \times 50^3}{36} + \frac{200 \times 50}{2} \times \left(\frac{350}{3}\right)^2$$

$$= 1.4768 \times 10^8 \text{ mm}^4$$

$$I_{YY} = \frac{100 \times 200^3}{3} + \frac{\pi \times 50^4}{8} + \frac{\pi \times 50^2}{2} \times (221.22)^2 + \frac{50 \times 200^3}{36} + \frac{50 \times 200}{2}\left(\frac{400}{3}\right)^3$$

$$= 5.613 \times 10^8 \text{ mm}^4$$

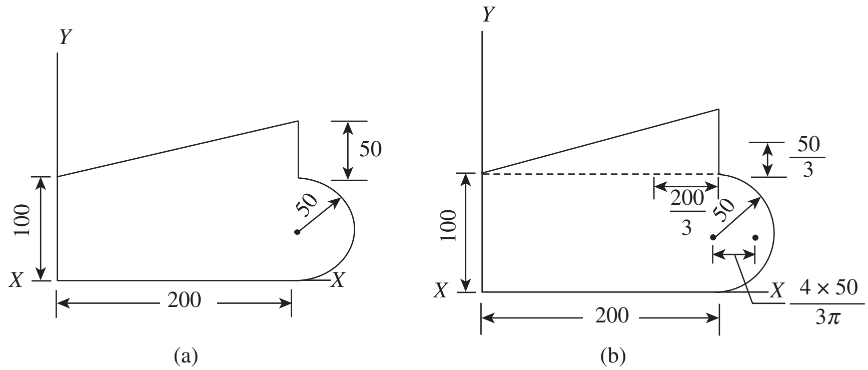


Fig. 1.42

1.9.5 Radius of Gyration

The radius of gyration is defined as $r = \sqrt{I/A}$, where I is the moment of inertia and A is the area of the section. This entity is frequently encountered in structural analysis and design.

The radius of gyration can be considered as the distance at which the whole area may be considered as concentrated as a strip, as shown in Fig. 1.43(a).

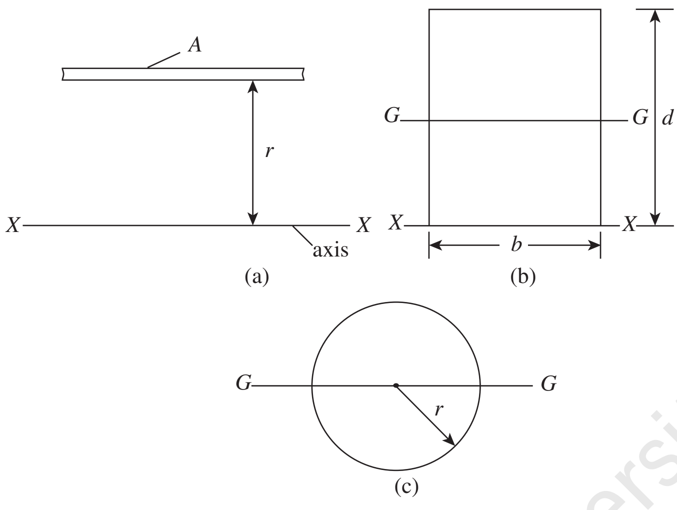


Fig. 1.43

Considering the rectangle shown in Fig. 1.43(b), the radius of gyration about the axis G-G is

$$r = \sqrt{\frac{I_{GG}}{A}} = \sqrt{\frac{bd^3}{12bd}} = \frac{d}{\sqrt{12}}$$

Similarly, for the circular area shown in Fig. 1.43(c),

$$R = \sqrt{\frac{I_{GG}}{A}} = \sqrt{\frac{\pi r^4}{4\pi r^2}} = \frac{r}{2}$$

In Fig. 1.43(b), $I_{XX} = I_{GG} + Ah^2$, where $h = d/2$. From this, $Ar^2_{XX} = Ar^2_{GG} + Ah^2$ or $r^2_{XX} = r^2_{GG} + h^2$. The polar radius of gyration, $r_z^{\,2} = r_x^{\,2} + r_y^{\,2}$.

The concept of radius of gyration finds application in the analysis and design of long columns. The following examples illustrate the procedure to compute the radius of gyration.

Example 1.16 Radius of gyration of square and hollow circular sections

Determine the radius of gyration of (i) a square section and box section and (ii) a hollow circular section shown in Fig. 1.44 about axes g-g and x-x.

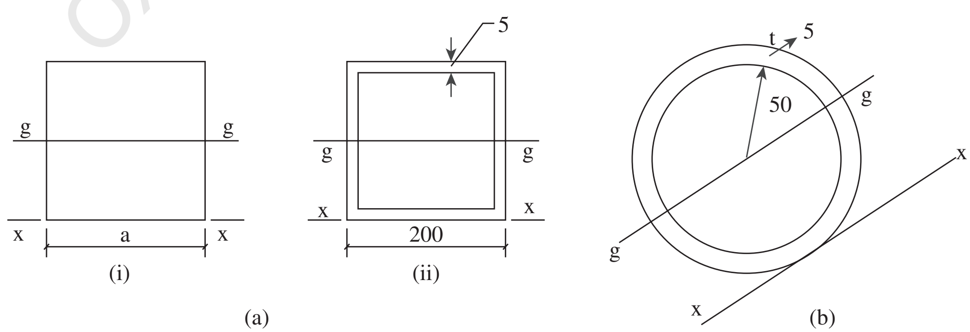


Fig. 1.44

Solution: (i) Square section: Area = a^2; $I_{gg} = a^4/12$; $I_{xx} = a^4/3$

$$r_{gg} = \sqrt{[(a^4/12)/a^2]} = a/\sqrt{12} = 0.29\,a;\ r_{xx} = \sqrt{[(a^4/3)/a^2]} = a/\sqrt{3} = 0.577\,a$$

For the box section given, Area = $200^2 - 190^2 = 3900\ \text{mm}^2$

$I_{gg} = [200^4 - 190^4]/12 = 24.73 \times 10^6$ mm^4; $I_{xx} = [200^4 - 190^4]/3 = 98.93 \times 10^6$ mm^4

Thus, $r_{gg} = \sqrt{[24.73 \times 10^6/3900]} = 79.63$ mm; $r_{xx} = \sqrt{[98.93 \times 10^6/3900]} = 159.27$ mm

(ii) In the case of the hollow circular section, inner radius = 50 – 5 = 45 mm

Area = $\pi(50^2 - 45^2) = 1492.25$ mm^2, $I_{gg} = (\pi/4)[50^4 - 45^4] = 1.688 \times 10^6$ mm^4

$I_{xx} = 1.688 \times 10^6 + 1492.25 \times 50^2 = 5.42 \times 10^6$ mm^4

Thus, $r_{gg} = \sqrt{[1.688 \times 10^6/1492.25]} = 33.6$ mm; $r_{xx} = \sqrt{[5.42 \times 10^6/1492.25]} = 60.3$ mm

Example 1.17 Radius of gyration of T-section

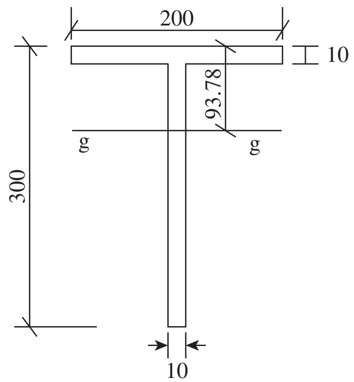


Fig. 1.45

Find the radius of gyration of the T-section about an axis passing through the centroid parallel to the flange (Fig 1.45).

Solution: Area of section = 200 × 10 + 290 × 10 = 4900 mm^2

We first locate the centroid of the section from the top,

$\ddot{y} = (2000 \times 5 + 2900 \times 155) / 4900 = 93.78$ mm

$I = 200 \times 10^3/12 + 2000 \times 88.78^2 + 10 \times 290^3/12 + 2900 \times 61.22^2$

$= 46.977 \times 10^6$ mm^4

Radius of gyration = $\sqrt{[46.977 \times 10^6/4900]} = 97.91$ mm

Example 1.18 Radius of gyration of channel section

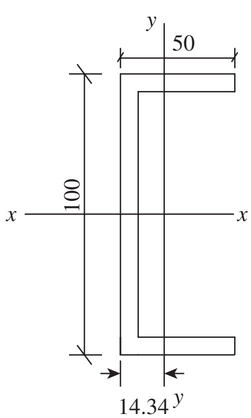


Fig. 1.46

Find the least radius of gyration of the channel section about axes parallel to the sides and passing through its centroid (Fig 1.46).

Solution: We locate the centroid and second moments of areas about X-X and Y-Y axes.

Area of section = 100 × 5 + 2 × 45 × 5 = 950 mm^2

Now, $\ddot{y} = [100 \times 5 \times 2.5 + 2 \times 45 \times 5 \times 27.5]/ 950 = 14.34$ mm

$I_{xx} = 5 \times 100^3/12 + 2[45 \times 5^3/12 + 45 \times 5 \times 47.5^2] = 416666.7 + 2[468.75 + 507656]$

$= 1.433 \times 10^6$ mm^4

Thus, $r_{xx} = \sqrt{[1.433 \times 10^6/950]} = 38.8$ mm

$I_{yy} = 100 \times 5^3/12 + 2[5 \times 45^3/12 + 5 \times 45 \times 13.16^2] = 225 \times 10^3$ mm^4

Thus $r_{yy} = \sqrt{[225 \times 10^3/950]} = 15.4$ mm

Least radius of gyration = 15.4 mm

1.10 SECTION MODULUS

Consider the beam shown in Fig. 1.47(a). When the beam is subjected to loads, it bends. The beam section is shown in Fig. 1.47(b). The centroid of the beam section can be found from the principles you might have studied in engineering mechanics. An axis through the centroid of the section and parallel to its base is called *neutral axis*. Note that this axis is perpendicular to the plane of the paper in Fig. 1.47(b). When the beam bends, each section of the beam rotates about the neutral axis. Because of this rotation, the MI (second moment of area) comes into effect in the bending of beams. This has been discussed in detail in the earlier sections.

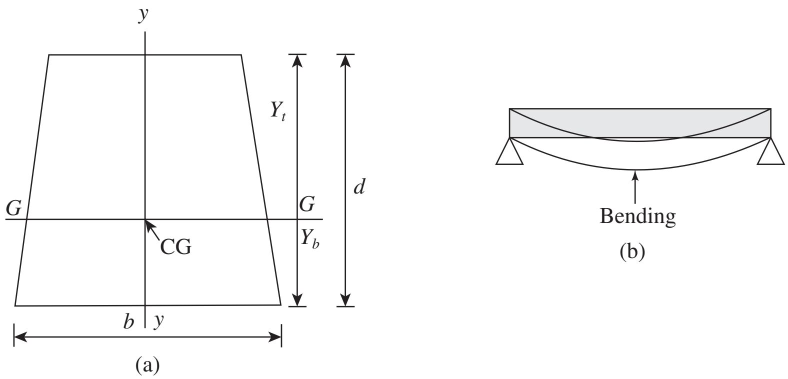


Fig. 1.47

The beam section is symmetrical about the *Y*-axis but can be unsymmetrical about the horizontal axis in the case of simple bending. The centroid, in such a case, will not be at the centre of the depth, *d*, but may be below or above. In the case shown in Fig. 1.47(b), the centroid is below the half depth *d*/2. The distances y_t and y_b thus are not equal. These distances are known as distances to the extreme fibres at top and bottom from the neutral axis (NA). The quantities I/y_t and I/y_b are known as moduli of section or section moduli of the beam section.

Section modulus is thus the MI of the section divided by the distance to the extreme fibre, either to the top or to the bottom, from the neutral axis. Section modulus is important because the least section modulus governs the design of the beam. Section modulus is usually denoted by the symbol *Z*. The unit of section modulus *Z* is L^3 being (L^4/L). The section modulus will have the units of mm^3, cm^3, m^3, etc.

We will illustrate the computation of section modulus through a number of examples. This will be useful while designing the beams.

Example 1.19 Section modulus

Compute the section modulus of the sections shown in Fig. 1.48.

Solution For the rectangular section shown in Fig. 1.48(a), the neutral axis will be at half depth as the section is symmetrical about that line. The neutral axis will be *g*-*g* at *d*/2 from the top fibre. The MI about the neutral axis is given by $bd^3/12$.

MI about neutral axis = $10 \times 20^3/12 = 6666.67\ cm^4$

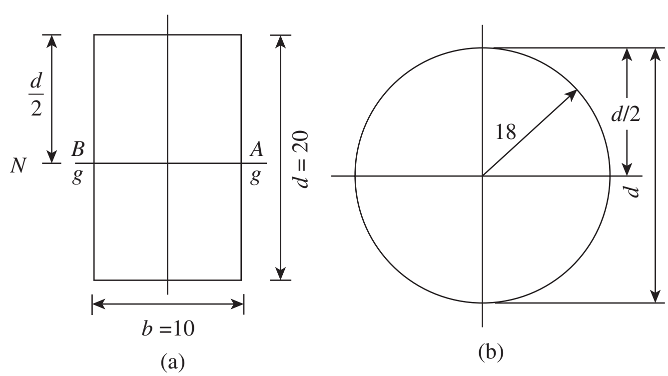


Fig. 1.48

In this case, y_t and y_b are equal, each equal to 10 cm.

Section modulus $Z = 6666.67/10 = 666.67$ cm^3 and is the same with respect to top and bottom fibres.

General formula in the case of rectangular section is $(bd^3/12)/(d/2) = bd^2/6$.

In the case of circular section shown in Fig. 1.48(b), the neutral axis will be at the centre of depth.

$$y_t = y_b = \text{radius} = 18 \text{ cm}$$

The neutral will coincide with a horizontal diameter.

$$\text{MI about NA} = \pi d^4/64 \text{ or } \pi r^4/4 = \pi(18)^4/4 = 82{,}448 \text{ cm}^4$$

Section modulus will be the same about top and bottom fibres.

$$\text{Section modulus} = 82{,}448/18 = 4580.44 \text{ cm}^3$$

The general formula for section modulus is $(\pi r^4/4)/r = \pi r^3/4 = \pi d^3/32$

Example 1.20 Section moduli of a semicircular section

A wooden semicircular log, of 20 cm radius, is used as a beam over a span of 2 m. Find the section moduli of the section.

Solution The section is shown in Fig. 1.49(a).

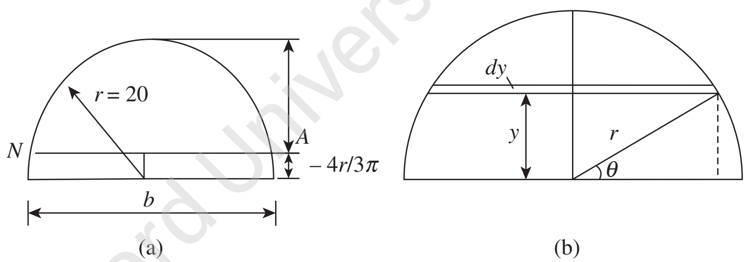


Fig. 1.49

The centroid of the semicircular section can be found by integration. We can use Table 1.1 of Appendix 1. The centroid is at $4r/3\pi$ from the base diameter. We will prove this by integration.

Area of the semi-circular area = $\pi r^2/2 = \pi(20)^2/2 = 628.32$ cm^2

We consider an elementary strip at y from the base. The width of the strip, $b = 2r\cos\theta$, $y = r\sin\theta$, and $dy = r\cos\theta\, d\theta$ [see Fig. 1.49(b)]

Moment of the elementary strip about the base = $(b\, dy)\, y$

$$\text{Moment} = (2r\cos\theta \times r\cos\theta\, d\theta)\, r\sin\theta$$
$$= 2r^3\cos^2\theta\sin\theta\, d\theta$$

Moment of the whole area is obtained by integrating from θ varying from 0 to $\pi/2$.

$$\text{Moment of the area} = \int_0^{\pi/2} [2r^3\cos^2\theta\sin\theta\, d\theta]$$
$$= 2r^3\left[-\cos^3\theta/3\right]_0^{\pi/2}$$
$$= 2r^3/3$$

Area of the section = $\pi r^2/2$

Distance of the centroid from the base y is given by

$$\hat{y} = (2r^3/3)/(\pi r^2/2)$$
$$= 4r/3\pi$$

Distance of the centroid from the base = 4 × 20/3π = 8.49 cm

The neutral axis will be as shown in Fig. 1.49(b).

To find the moment of inertia, we find $(b\,dy)y^2$ and integrate the same from 0 to π/2.

Second moment of area of the elementary strip = $(2r\cos\theta \times r\cos\theta\, d\theta)(r\sin\theta)^2$

$$= 2r^4\cos^2\theta\sin^2\theta\, d\theta$$

$$\text{MI} = \int_0^{\pi/2} (2r^4\cos^2\theta\sin^2\theta)\, d\theta$$

$$= 2r^4 \int_0^{\pi/2} \sin^2\theta(1-\sin^2\theta)\, d\theta$$

$$= 2r^4 \left[(\theta/2 - \sin 2\theta/4) + (1/4)\cos\theta\sin^3\theta - (3/4)\int \sin^2\theta\, d\theta\right]_0^{\pi/2}$$

$$= 0.11\, r^4$$

$$= 17{,}600 \text{ cm}^4$$

$$Z_t = 17{,}600/11.51 = 1529 \text{ cm}^3$$

$$Z_b = 17{,}600/8.49 = 2073 \text{ cm}^3$$

Example 1.21 Section moduli of a triangular section

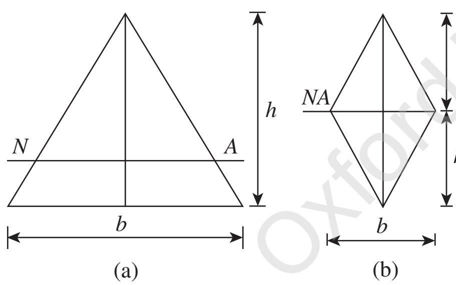


Fig. 1.50

Find the section moduli of an isosceles triangle. If two such triangles were joined base to base, what will be the section modulus of the figure? It is given that b = 30 cm and h = 18 cm.

Solution The isosceles triangle is shown in Fig. 1.50(a). The centroid of the triangle is at $h/3$ from the base. In this case, it is at 6 cm from the base.

MI of the triangle about the NA = $bh^3/36 = 30 \times 18^3/36 = 4860$ cm^4

As the section is unsymmetrical about the NA, there will be two section moduli.

(based on top edge) $Z_t = 4860/(2 \times 18/3) = 405$ cm^3

(based on bottom edge) $Z_b = 4860/6 = 810$ cm^3

If two such triangles were joined together as shown in Fig. 1.50(b),

MI about NA = $2 \times bh^3/12 = 2 \times 30 \times 18^3/12 = 29{,}160$ cm^4

Section modulus will be the same about top and bottom edges.

$Z = 29{,}160/18 = 1620$ cm^3

Example 1.22 Section modulus of I-sections

Find the section modulus of the I-sections shown in Fig. 1.51

Solution The I-section shown in Fig. 1.51(a) is a symmetrical section, symmetrical about the X–X axis. The section modulus with respect to top and bottom fibres will be equal.

We first find the second moment of area of the section about the neutral axis (X–X axis).

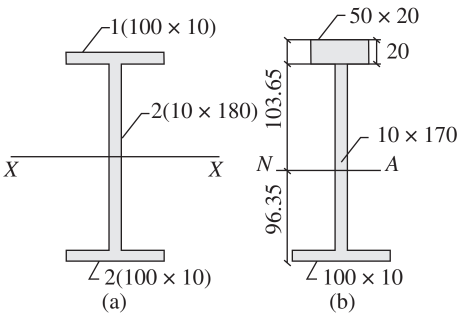


Fig. 1.51

The second moment of area is the second moment of area of the three rectangles marked 1 (100 × 10), 2(180 × 10) and 3(100 × 10). As the two flanges are placed symmetrically about NA, their MI about *X-X* will be equal. The MI can be calculated as

$$I_{XX} - 2\,[100 \times 10^3/12 + 100 \times 10 \times 95^2] + 10 \times 180^3/12$$
$$= 2[10^5/12 + 9.025 \times 10^6] + 4.86 \times 10^6 = 4.868 \times 10^6 \text{ mm}^4.$$

Section modulus $Z_t = Z_b = 4.868 \times 10^6/100 = 4.868 \times 10^4$ mm^4.

The I-section shown in Fig. 1.51 (b) is not symmetrical about the horizontal axis.
We locate the centroid of the section.

Area of the section = 100 × 10 + 170 × 10 + 50 × 20 = 3700 mm^2.

$$\hat{Y} = [100 \times 10 \times 5 + 170 \times 10 \times 95 + 50 \times 20 \times 190]/3700 = 96.35 \text{ mm from bottom.}$$

The distance of NA from the top = 103.65 mm.

$$I_{XX} = [100 \times 10^3/12 + 1000 \times 91.35^2 + 10 \times 170^3/12 + 1700 \times 1.35^2 + 50 \times 20^3/12 + 1000 \times 93.65^2]$$
$$= 21.26 \times 10^6 \text{ mm}^4$$

Section modulus with respect to top $Z_t = 21.25 \times 10^6/103.65 = 205017$ mm^3.
Section modulus with respect to bottom $Z_b = 21.25 \times 10^6/96.35 = 220550$ mm^3.

Example 1.23 Section modulus of a trapezoidal section

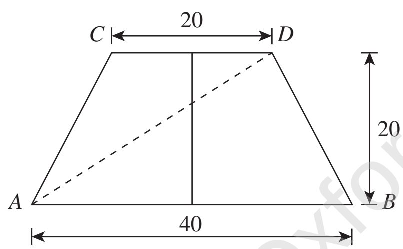


Fig. 1.52

Find the section modulus of the trapezium shown in Fig. 1.52.

Solution We first find the centroid of the section. Taking moments about the bottom edge, considering two triangles as shown,

Area, $\bar{y}$ = [20 × (20/2) (2 × 20/3) + 40 × (20/2) (20/3)] = 5333.33 cm^3

Area of trapezium = [(40 +20)/2] × 20 = 600 cm^2

$\hat{y}$ = 5333.33/600 = 8.89 cm

Height of centroid from top = 20 – 8.89 = 11.11 cm

$$\text{MI about NA} = 40 \times 20^3/12 + (40 \times 20/2)\,(8.89 - 6.67)^2$$
$$+ 20 \times 20^3/12 + 20 \times (20/2)\,(11.11 - 6.67)^2$$
$$= 26{,}666.67 + 1971.36 + 13{,}333.33 + 3942.72$$
$$= 45{,}914 \text{ cm}^4$$
$$Z_t = 45{,}914/11.11 = 4132.68 \text{ cm}^3$$
$$Z_b = 45{,}914/8.89 = 514.68 \text{ cm}^3$$

Example 1.24 Section modulii of two angle sections

Two unequal angle sections are kept as shown in Fig. 1.53. Find the distance *x* so that the MI about *X-X* and *Y-Y* axes are equal. Also find the section modulii.

Solution We first locate $\hat{Y}$, the distance of centroid from the top flange.

$$\hat{Y} = [60 \times 10 \times 5 + 110 \times 10 \times 65]/(60 \times 10 + 110 \times 10) = 43.82 \text{ mm}$$

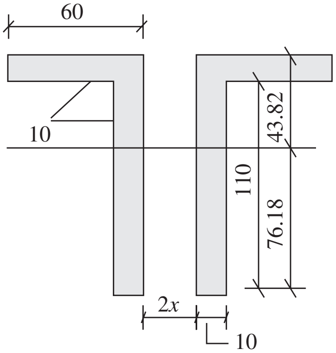


Fig 1.53

Distance of centroid from bottom fibre = 120 – 43.82 = 76.18 mm.

MI about X-X axis, I_{XX}, is given by (for one angle section)

$$I_{XX} = 60 \times 10^3/12 + 600 \times 38.82^2 + 10 \times 110^3/12 + 1100 \times 21.18^2$$
$$= 2479500 \text{ mm}^4$$

I_{XX} for the whole section = 2 × 2479500 = 4959000 mm^4.

Let $2x$ be the clear distance between the angles.

$$I_{YY} = 10 \times 60^3/12 + 600\,(30 + x)^2 + 110 \times 10^3/12 + 1100\,(5 + x)^2$$
$$= 1700\,x^2 + 47000x + 756667 \text{ (for one angle)}$$

I_{YY} for two angles will be double of this due to symmetry about Y-Y axis.

As $I_{XX} = I_{YY}$, we get, $2[1700x^2 + 47000x + 756667 = 4959000$.

Reducing this expression, $x^2 + 27.64x - 1013.4 = 0$; $x = 20.88$ mm

Distance between channels = 41.76 mm.

Section modulus Z_t = 4959000/43.82 = 113168 mm^3.

Section modulus Z_b = 4959000/76.18 = 65096 mm^3.

Example 1.25 Section modulus of unequal channel section

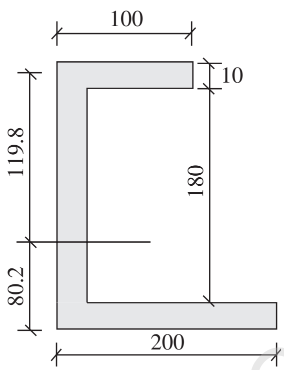


Fig 1.54

Find the section modulii of the unequal channel section shown in Fig. 1.54.

Solution As the flange lengths are unequal, we have to locate centroid.

Area of the section = 100 × 10 + 180 × 10 + 200 × 10 = 4800 mm^2.

$\hat{Y}$ = [100 × 10 × 5 + 180 × 10 × 100 + 200 × 10 × 195] /4800 = 119.8 mm

Distance to the NA from bottom = 80.20 mm

$$I_{XX} = [100 \times 10^3/12 + 1000 \times 114.8^2 + 10 \times 180^3/12 + 1800 \times 19.8^2 + 200 \times 10^3/12 + 2000 \times 75.20^3]$$
$$= 30.1 \times 10^6 \text{ mm}^4.$$

Section modulus Z_t = 30.1 × 106/119.8 = 251089 mm^3.

Section modulus Z_b = 30.1 × 106 / 80.2 = 375068.7 mm^4.

Example 1.26 Section modulus of a given section

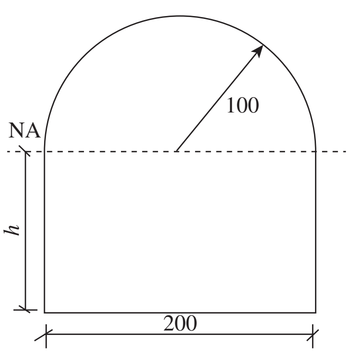


Fig. 1.55

The section shown in Fig. 1.55 consists of a rectangle and a semicircle. The height of the rectangle h is so adjusted that the NA of the section falls at the junction of the two parts. Find the section modulus of the section.

Solution Area of semicircle = $\pi \times 100^2/2$ = 15708 mm^2; Area of rectangle = $200h$ mm^2. Centroid of semicircle = $4x\ 100/3\pi$ = 42.44 mm from base. Taking moments about the base of the section, we get

$$\hat{Y} = [15708 \times 42.44 + 200\,h^2/2]\,/(15708 + 220\,h) = h;$$

from this we get h = 81.65 mm

$$I_{NA} = [\pi\,(100)^4/8 + 200 \times 81.65^3/3] = 48.34 \times 10^6 \text{ mm}^4.$$

Section modulus at top, $Z_t = 48.34 \times 10^6/100 = 48.34 \times 10^4$ mm^3.

Section modulus at bottom = $48.34 \times 10^6/81.65 = 59.2 \times 10^4$ mm^3.

1.11 PRODUCT OF INERTIA

The product of inertia, usually denoted by the symbol P, is given by the mathematical expression $\int_A xydA$. Though the concept does not find wide application, it is useful in dealing with unsymmetrical bending, finding principal axes of inertia, in advanced structural analysis, etc.

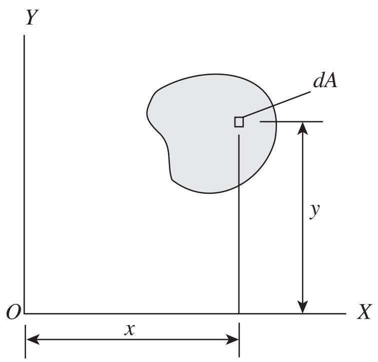


Fig. 1.56

Consider Fig. 1.56. The product of inertia can be expressed as $P = \int_A xydA$. The unit of P is the same as that of I, i.e., mm^4, m^4, etc. But while I is always positive, P can be positive or negative, depending upon the axes chosen. Note that P has to be with reference a set of coordinate axes and hence should be expressed as $P_{XY} = \int_A xydA$.

Let us take the case of a rectangle and the set of axes $X_1 - Y_1$, $X_2 - Y_2$, $X_3 - Y_3$, etc. as in Fig. 1.57. With respect to the set of axes X_1,Y_1, $P_{x_1y_1} = \int_A x_1y_1dA$. Since X_1, Y_1 are always positive, $P_{x_1y_1}$ is always positive. With respect to the axes X_2, Y_2, note that X_2 and Y_2 are axes of symmetry. For every elementary area dA with positive x-coordinates, there is a corresponding area with negative x-coordinates with respect to the Y-axis, and for every elementary area with positive y-coordinates, there is a corresponding area with negative y-coordinates with respect to the X-axis. Thus, $P_{x_2y_2} = \int_A x_{22}dA = 0$.

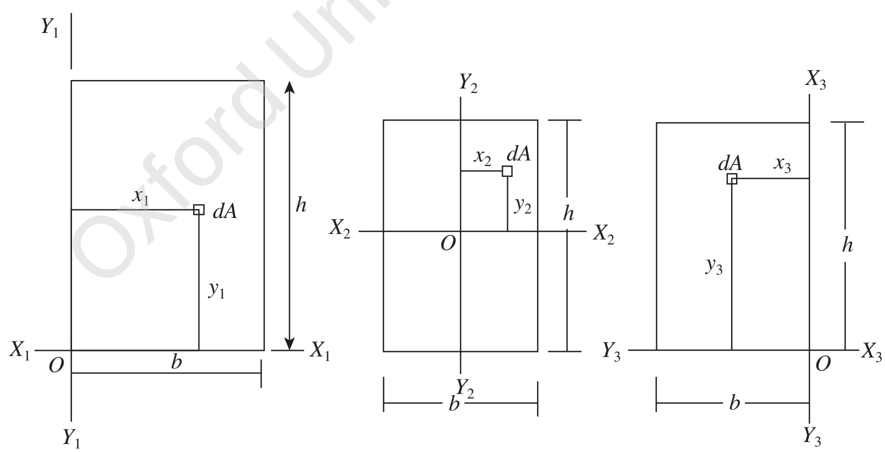


Fig. 1.57

With respect to axes x_3, y_3, the area is so located that the y_3-coordinates are positive while the x_3-coordinates are negative. Therefore, $Px_3y_3 = \int_A x_3y_3dA$ is negative.

An important property of product of inertia is that it is zero with respect to a set of axes if one or both of the axes are axes of symmetry.

Thus, in Fig. 1.58, $P_{XY} = 0$ for the T-section because the Y-axis is an axis of symmetry and $P_{XY} = 0$ for a circular area about the axes passing through its centre.

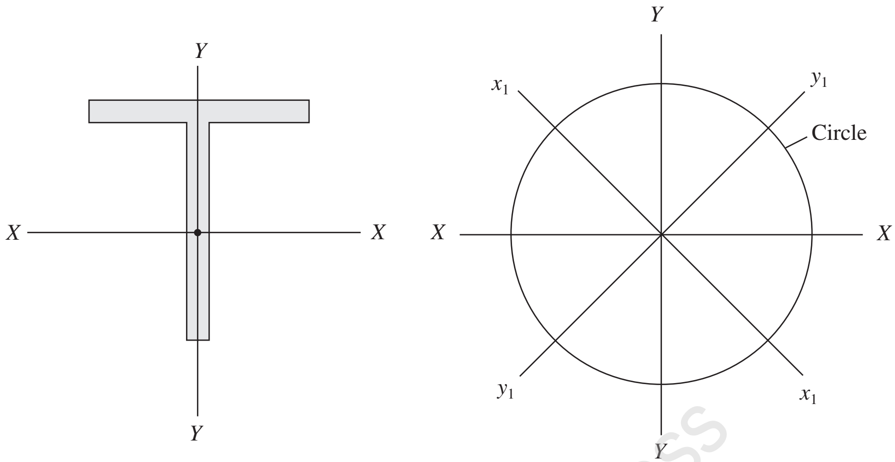


Fig. 1.58

Example 1.27 PI of a rectangle

Find the product of inertia of the rectangle with reference to the X, Y axes shown in Fig. 1.59.

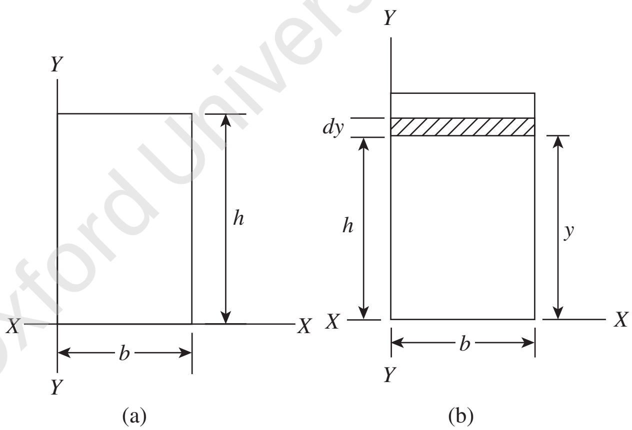


Fig. 1.59

Solution Consider a strip parallel to the X-axis as shown in Fig. 1.59(b). Note that (x, y) are the coordinates of the centroid of the strip. From the figure, $dA = b\,dy$, $\overline{x} = b/2$, $\overline{y} = y$. Therefore,

$$P_{XY} = \int_A xy\,dA = \int_0^h \frac{b}{2} yb\,dy = \frac{b^2h^2}{4}$$

Example 1.28 PI of a right-angled triangle (general formula)

Find the product of inertia of the right-angled triangle shown in Fig. 1.60 with respect to the X, Y axes.

Solution The whole triangle is in the first quadrant. The x and y coordinates are positive throughout. Consider the elementary strip parallel to the base, as shown in Fig. 1.60(b).

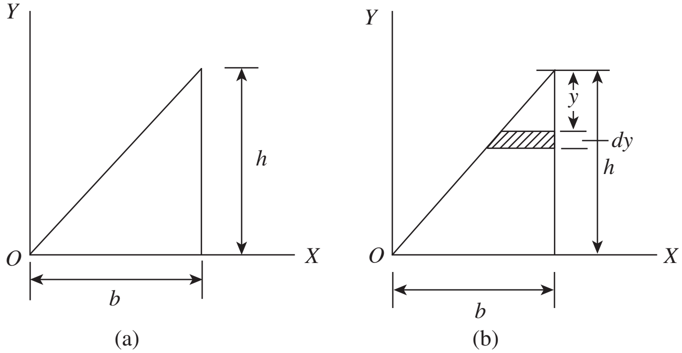


Fig. 1.60

Width of the strip $= b\,\dfrac{y}{h}$

Area $= b\,\dfrac{y}{h}\,dy$

x-coordinate of centroid $= b - \dfrac{1}{2} b \dfrac{y}{h}$

$$= \frac{b}{2h}(2h - y)$$

y-coordinate $= (h - y)$

Therefore,

$$P_{XY} = \int_A xy\,dA = \int_0^h \frac{b}{2h}(2h - y)(h - y)dy\, b\frac{y}{h}$$

$$= \int_0^h \frac{b^2}{2h^2}(2h^2 - 2hy - yh + y)ydy$$

$$= \int_0^h \frac{b^2}{2h^2}(2h^2y - 3hy^2 + y^3)dy$$

$$= \frac{b^2}{2h^2}\left[\frac{2h^2y^2}{2} - \frac{3hy^3}{3} + \frac{y^4}{4}\right]_0^h = \frac{b^2}{2h^2}\frac{h^4}{4} = \frac{b^2h^2}{8}$$

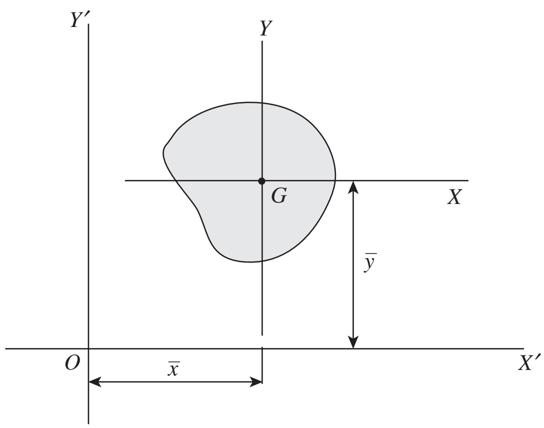


Fig. 1.61

Transfer of axes for product of inertia In Fig. 1.61, G is the centroid of the area shown and P_{XY} is the product of inertia with respect to the axes X, Y through G. The product of inertia of this area with reference to axes X', Y' through point O, such that the coordinates of G (with respect to axes X', Y') are $(\bar{x}, \bar{y})$, is given by

$P_{X'Y'} = \int_A (x + \bar{x})(y + \bar{y})\,dA$

$= \int_A xy\,dA + \int_A x\bar{y}\,dA + \int_A \bar{x}\,y\,dA + \int_A \bar{x}\,\bar{y}\,dA$

Note that $(\bar{x}, \bar{y})$ are constants, being the coordinates of G. $\int_A \bar{y}\,x\,dA$ and $\int_A \bar{x}\,y\,dA$ are equal to zero, since they are the moments of the area about its centroid. $\int_A \bar{x}\,\bar{y}$, dA is $A\,\bar{x}\,\bar{y}$ and $\int_A xy\,dA = P_{XY}$. Therefore,

$P_{X'Y'} = P_{XY} + A\,\bar{x}\,\bar{y}$

That is, *the product of inertia of an area with respect to a parallel set of axes is equal to the sum of the product of inertia with respect to parallel axes through the centroid and the product of the area and coordinates of the centroid with respect to the new set of axes*. This result is useful in finding the product of inertia of composite areas.

Example 1.29 PI of an angle section

Find the product of inertia of the equal angle section shown, about the axes passing through its centroid and the axes passing through its edges (Fig. 1.62).

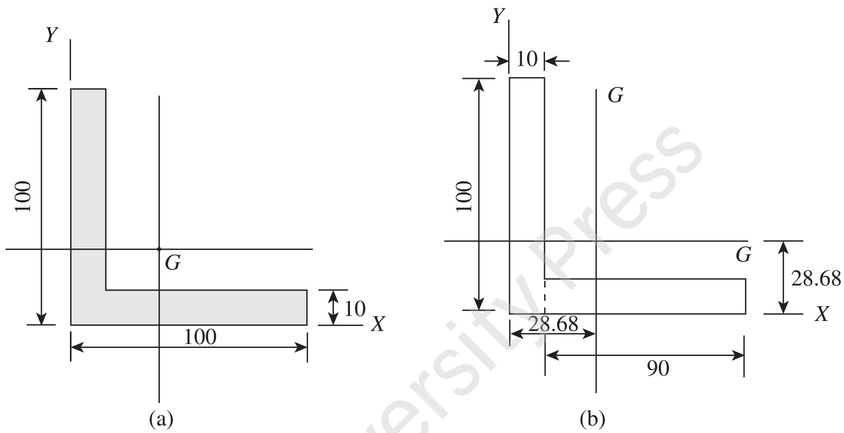


Fig. 1.62

Solution We first determine the coordinates of the centroid of the section. $\bar{x} = \bar{y}$ as the section is an equal angle section.

$$(100 \times 10 + 90 \times 10)\ \bar{x} = 100 \times 10 \times 5 + 90 \times 10 \times 55,\ \bar{x} = 28.68 \text{ mm}$$

$$(100 \times 10 + 90 \times 10)\ \bar{y} = 100 \times 10 \times 50 + 90 \times 10 \times 5,\ \bar{y} = 28.68 \text{ mm}$$

The section consists of two rectangles, 100 × 10 and 90 × 10. The centroidal axes of both the rectangles are symmetrical and the product of inertia about their own centroidal axes is zero.

From Fig. 1.62(b),

$$P_{XY} = A_1\ \bar{x}\ \bar{y}\ + A_2\ \bar{x}\ \bar{y}$$

$$= 100 \times 10 \times 5 \times 50 + 90 \times 10 \times 55 \times 5$$

$$= 497.5 \times 10^3 \text{ mm}^4$$

$$P_{GG} = 100 \times 10 \times 23.68 \times (-21.32) + 900 \times 23.68 \times (-26.32)$$

$$= -10657 \times 10^3 \text{ mm}^4$$

Example 1.30 PI of a right-angled triangle

Find the product of inertia of the right-angled triangle shown in Fig. 1.63(a) about the *X*- and *Y*-axes passing through its centroid, and about the *X*′- and *Y*′-axes passing through its sides.

Solution From Fig. 1.63(b), we find *PX*′Y′ by integration.

$$\text{Width of elementary strip} = \frac{90}{120} y = \frac{3}{4} y$$

$$\text{Area} = \frac{3}{4} y\, dy$$

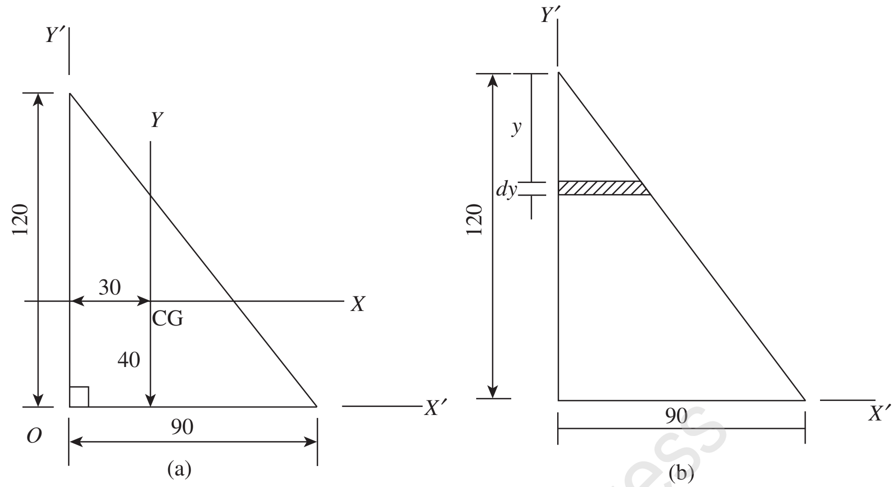


Fig. 1.63

Coordinates of it centroid = $\left(\frac{3}{8}y, (120 - y)\right)$

$$P_{X'Y'} = \int_0^{120} \frac{3}{4} y\, dy \frac{3}{8} y\,(120 - y)$$

$$= \frac{9}{32}\int_0^{120} (120y^2 - y^3)\, dy$$

$$= \frac{1}{12} \times \frac{9}{32} (120)^4 = 4.86 \times 10^6 \text{ mm}^4$$

P_{XY} (for centroidal axes) can be found from the transfer formula:

$$P_{X'Y'} = P_{XY} + A\ \bar{x}\ \bar{y}$$

$$P_{X'Y'} = 4.86 \times 10^6, \quad A = \frac{120 \times 90}{2} = 5400, \quad \bar{x} = 30, \quad \bar{y} = 40$$

$$P_{XY} = 4.86 \times 10^6 - 5400 \times 30 \times 40$$

$$= -1.62 \times 10^6 \text{ mm}^4$$

To derive a general formula for this case, assume the base as 'b' and height as 'h' for the triangle.

Area of the elementary strip = bdy/h.

Coordinates of the centroid of the strip are $[by/2h, (h - y)]$

Product of inertia of the strip about axes X', Y' is given by

$$P_{X',Y'} = A \times y = (bdy/h)\,(by/2h)\,(h - y).$$

To get the product of inertia for the whole area, integrate from 0 to h.

$$P_{X',Y'} = \int_0^h \left[(by\, dy/h)\,(by/2h)\,(h - y)\right]$$

$$= (b^2/2h^2) \int_0^h (hy^2 - y^3)\ dy = (by^2/2h^2)\left[hy^3/3 - y^4/4\right]_0^h = b^2h^2/24$$

To find the product of inertia about the centroidal axes, note that the coordinates of the centroid are $(b/3, h/3)$ with respect to $X' - Y'$ axes.

From the transfer formula, $P_{X'Y'} = P_{XY} + Axy$, we get

$$P_{X,Y} = P_{X'Y'} - Axy = (b^2h^2/24) - (bh/2)(b/3)(h/3)$$
$$= (b^2h^2/24) - (b^2h^2/18) = -(b^2h^2/72).$$

In the present case, $b = 90$ mm and $h = 120$ mm

$$P_{X'Y'} = 90^2 \times 120^2/24 = 4.86 \times 10^6 \text{ mm}^4.$$
$$P_{X,Y} = -(90^2 \times 120^2/72) = -1.62 \times 10^6 \text{ mm}^4, \text{ as before.}$$

Example 1.31 Product of inertia of channel section

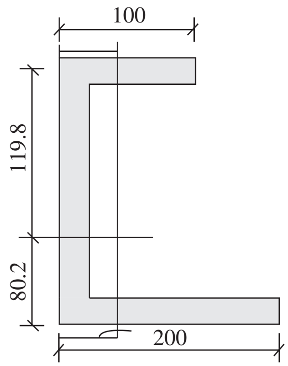


Fig. 1.64

Find the product of inertia of the unequal channel section about axes passing through its centroid.

Solution We first locate the centroid of the section.

Area of the section = 100 × 10 + 180 × 10 + 200 × 10 = 4800 mm^2.

$\hat{Y}$ = [100 × 10 × 5 + 180 × 10 × 100 + 200 × 10 × 195] /4800 = 575000/4800
= 119.8 mm

X = [100 × 10 × 50 + 180 × 10 × 5 + 200 × 10 × 100]/4800 = 214000/4800
= 44.58 mm

The centroidal axes are as shown in figure.

To find the product of inertia we consider the area as consisting of three rectangles marked 1, 2, and 3. The product of inertia of the rectangles about their own centroidal axes is zero. The calculations are done in the table below:

No.	Area	X	Y	A × Y
1.	100 × 10 = 1000 mm^2	5.42	114.8	622216 mm^4
2.	180 × 10 = 1800 mm^2	–39.58	19.8	–1410631 mm^4
3.	200 × 10 = 2000 mm^2	55.42	–75.20	–8335168 mm^4

Product of inertia, $P_{x,y} = -9123583$ mm^4.

Example 1.32 Product of inertia of composite section

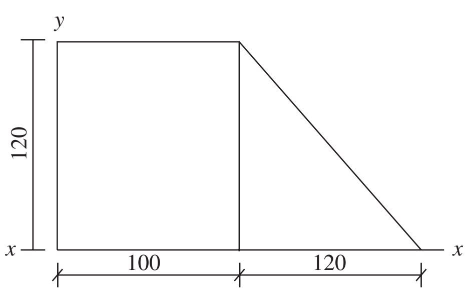


Fig 1.65

Determine the product of inertia of the composite section, shown in Fig. 1.65 consisting of a rectangle and a right triangle about axes X-X and Y-Y.

Solution We note that the area consists of a rectangle, a symmetrical figure, and a triangle which is not symmetrical. Thus in the case of triangle the product of inertia about its own centroidal axes is not zero.

The product of inertia of the triangle can be found from integration. We take the value from Example 1.30.

Product of inertia about centroidal axes = $-bh^2/72 = -120^2 \times 120^2/72 = -2.88 \times 10^6$ mm^4

The calculations can be done as in the table below:

Area	***X***	***Y***	***P about centroidal axes***	***Product of inertia***
Rectangle 100 × 120 Area = 12000 mm^2	50	60	Zero	12000 × 50 × 60 = 36 × 10^6 mm^4
Triangle 120 × 120/2 Area = 7200 mm^2	140	40	2.88 × 10^6 mm^4	7200 × 140 × 40 = 40.32 × 10^6 mm^4

Product of inertia $P_{xy} = (-2.88 + 36 + 40.32) \times 10^6 = 73.44 \times 10^6$ mm^4.

1.12 PRINCIPAL AXES FOR MI

We have seen that an area has a moment of inertia and a product of inertia about the axes passing through any point. Considering the area shown in Fig. 1.65(a), I_{XX}, I_{YY}, and P_{XY} can be calculated about the axes *X*-*X* and *Y*-*Y* passing through any point *O*.

If we consider any two mutually perpendicular axes *U* and *V* which are inclined at an angle θ to the axes *X* and *Y* and passing through *O*, it is possible to derive a relationship between I_{UU}, I_{VV}, and P_{UV} and the quantities I_{XX}, I_{YY}, and P_{XY}. Such a relationship has some important applications in structural mechanics.

From Fig. 1.66(b), we observe that point *P* has coordinates (*x*, *y*) with respect to the *X*, *Y* axes. The coordinates of point *P* with respect to axes *U* and *V* are *u* and *v* such that

$$v = y\cos\theta - x\sin\theta$$
$$u = y\sin\theta + x\cos\theta$$

I_{UU} and I_{VV} can be defined by the integrals $\int u^2 dA$ and $\int v^2 dA$. Thus,

$$I_{VV} = \int v^2 dA = \int (y\cos\theta - x\sin\theta)^2\, dA$$

$$= \int y^2\cos^2\theta\, dA + \int x^2\sin^2\theta\, dA - \int 2xy\sin\theta\cos\theta\, dA$$

$$\int y^2\cos^2\theta\, dA = I_{XX}\cos^2\theta; \int x^2\sin^2\theta\, dA = I_{YY}\sin^2\theta$$

and

$$\int 2xy\sin\theta\cos\theta\, dA = 2P_{XY}\sin\theta\cos\theta$$

$$I_{VV} = I_{XX}\cos^2\theta + I_{YY}\sin^2\theta - 2P_{XY}\sin\theta\cos\theta$$

Similarly,

$$I_{UU} = \int u^2\, dA = \int (y\sin\theta + x\cos\theta)^2\, dA$$

$$= \int y^2\sin^2\theta\, dA + \int x^2\cos^2\theta\, dA + \int 2xy\sin\theta\cos\theta\, dA$$

$$= I_{XX}\sin^2\theta + I_{YY}\cos^2\theta + 2P_{XY}\sin\theta\cos\theta$$

Since $\cos^2\theta = \left(\frac{1+\cos 2\theta}{2}\right)$, $\sin^2\theta = \left(\frac{1-\cos 2\theta}{2}\right)$, and $2\sin\theta\cos\theta = \sin 2\theta$

$$I_{VV} = I_{XX}\left(\frac{1+\cos 2\theta}{2}\right) + I_{YY}\left(\frac{1-\cos 2\theta}{2}\right) + P_{XY}\sin 2\theta$$

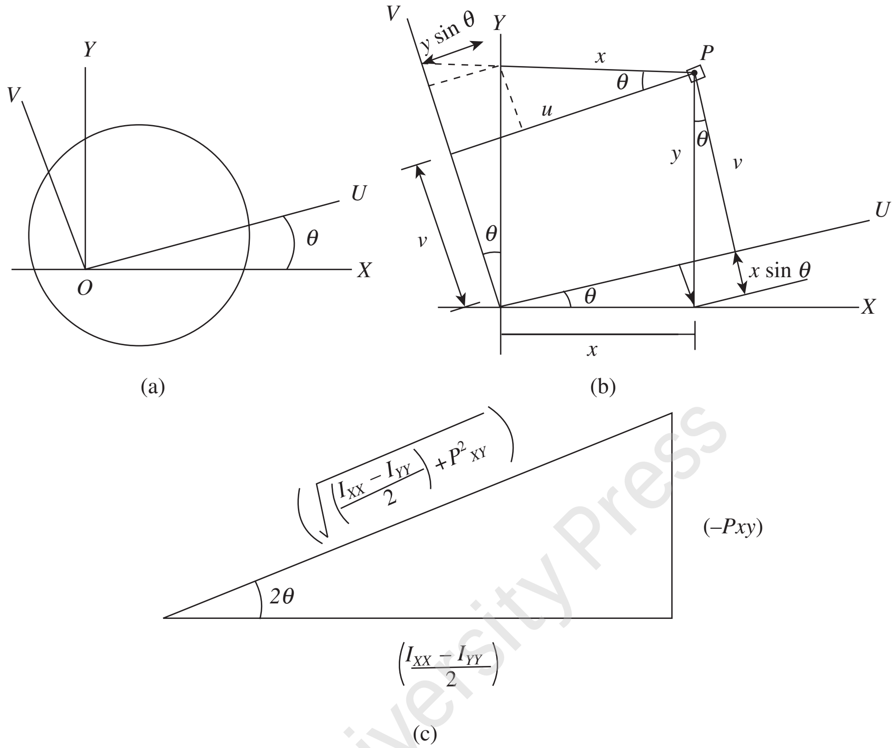


Fig. 1.66

$$= \frac{I_{XX} + I_{YY}}{2} + \frac{I_{XX} - I_{YY}}{2} \cos 2\theta + P_{XY} \sin 2\theta$$

$$I_{UU} = I_{XX}\left(\frac{1-\cos 2\theta}{2}\right) + I_{YY}\left(\frac{1+\cos 2\theta}{2}\right) + P_{XY} \sin 2\theta$$

$$= \left(\frac{I_{XX} + I_{YY}}{2}\right) - \left(\frac{I_{XX} - I_{YY}}{2}\right) \cos 2\theta + P_{XY} \sin 2\theta$$

Product of inertia $P_{UV} = \int uv dA$

$$= \int (y \cos\theta - x \sin\theta)(y \sin\theta + x\cos\theta)\, dA$$

$$= \int (y^2 \sin\theta \cos\theta + xy \cos^2\theta - xy \sin^2\theta - x^2 \sin\theta \cos\theta)\, dA$$

$$= I_{XX} \sin\theta \cos\theta + P_{XY} \cos^2\theta - P_{XY} \sin^2\theta - I_{YY} \sin\theta \cos\theta$$

$$= \frac{I_{XX} - I_{YY}}{2} \sin 2\theta + P_{XY} \cos 2\theta \quad [\because \cos^2\theta - \sin^2\theta = \cos 2\theta]$$

As the angle θ changes, we get different orientations of the U- and V-axis. I_{UU}, I_{VV}, and P_{UV} change with values of θ. To get the maximum values of I_{UU} and I_{VV}, we differentiate with respect to θ and equate the results to zero to find the value of θ, and then substitute this value of θ in the expressions for I_{UU} and I_{VV}. $d(I_{UU})/d\theta = 0$ gives

$$\frac{I_{XX} - I_{YY}}{2}(-2 \sin 2\theta) + P_{XY}(-2 \cos 2\theta) = 0$$

which gives

$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{-P_{XY}}{\left(\dfrac{I_{XX} - I_{YY}}{2}\right)}$$

$d(I_{VV})/d\theta = 0$ gives

$$\frac{I_{XX} - I_{YY}}{2} \times 2\sin 2\theta + P_{XY} \times 2\cos 2\theta = 0$$

Please note that the differentiation of I_{UU} and I_{VV} with respect to θ gives an expression which is the value of P_{XY}. This means that when the maximum or minimum value of I_{UU} and I_{VV} is obtained, P_{XY} is zero.

For the value of θ, we have the expression

$$\tan 2\theta = \frac{-P_{XY}}{\left(\dfrac{I_{XX} - I_{YY}}{2}\right)}$$

From Fig. 1.66(c), we get

$$\sin 2\theta = \frac{-P_{XY}}{\sqrt{\left(\dfrac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}}$$

$$\cos 2\theta = \frac{\left(\dfrac{I_{XX} - I_{YY}}{2}\right)}{\sqrt{\left(\dfrac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}}$$

These values can be substituted in the expressions for I_{VV} and I_{UU}. Substituting,

$$I_{VV} = \frac{I_{XX} + I_{YY}}{2} + \frac{\left(\dfrac{I_{XX} - I_{YY}}{2}\right)\left(\dfrac{I_{XX} - I_{YY}}{2}\right)}{\sqrt{\left(\dfrac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}} - \frac{P_{XY}\left(-P_{XY}\right)}{\sqrt{\left(\dfrac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}}$$

$$= \frac{I_{XX} + I_{YY}}{2} - \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}$$

$$I_{UU} = \frac{I_{XX} + I_{YY}}{2} - \frac{\left(\dfrac{I_{XX} - I_{YY}}{2}\right)\left(\dfrac{I_{XX} - I_{YY}}{2}\right)}{\sqrt{\left(\dfrac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}} + \frac{P_{XY}\left(-P_{XY}\right)}{\sqrt{\left(\dfrac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}}$$

$$= \frac{I_{XX} + I_{YY}}{2} \pm \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}$$

The maximum (or minimum) values of I are known as principal moments of inertia and the axes are known as principal axes. Also note that

$$I_{UU} + I_{VV} = I_{XX} + I_{YY} = J, \text{ the polar MI}$$

The principal moment of inertia can be expressed as

$$I_{1,2} = \frac{I_{XX} + I_{YY}}{2} \pm \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}$$

The orientation of the principal axes is given by

$$\tan 2\theta = \frac{-P_{XY}}{\left(\frac{I_{XX} - I_{YY}}{2}\right)}$$

The product of inertia P_{XY} is zero about the principal axes.

The following points can easily be seen from the equations derived above.

(i) $I_{UU} + I_{VV} = I_{XX} + I_{YY}$ = constant (this constant is, of course, the polar MI J).

(ii) There will be a set of axes in any area, through its centroid, which are the principal axes. The MI is maximum about one of these and minimum about the other.

(iii) The product of inertia is zero about the principal axes.

(iv) Axes of symmetry of an area are principal axes. (The principal axes need not be the axes of symmetry.)

Example 1.33 Principal axes and principal MI of a triangle

For the right-angled triangle shown in Fig. 1.67(a), determine the principal axes through the centroid and the MI about these axes.

Solution Let *X-X* and *Y-Y* be two mutually perpendicular axes through the centroid. From Fig. 1.67(b), considering the elementary strip shown,

$$I_{XX} = \int_0^{120} \frac{3}{4}\, y\, dy\, (80 - y)^2 = \frac{3}{4}\int_0^{120} y\,(6400 + y^2 - 160y)dy$$

$$= \frac{3}{4}\left[\frac{6400y^2}{2} + \frac{y^4}{4} - \frac{160y^3}{3}\right]_0^{120}$$

$$= \frac{3}{4} \times 120 \times 120 \left[3200 + \frac{120\times120}{4} - \frac{160\times120}{3}\right]$$

$$= 4.32 \times 10^6 \text{ mm}^4$$

$$I_{YY} = \int_0^{90} \frac{4}{3}\, x dx\, (60 - x)^2 = \frac{4}{3}\int_0^{90} x\,(3600 + x^2 - 120x)dx$$

$$= \frac{4}{3}\left[\frac{3600x^2}{2} + \frac{x^4}{4} - \frac{120x^3}{3}\right]_0^{90}$$

$$= \frac{4}{3}\times90\times90\left[1800 + \frac{90\times90}{4} - 40\times90\right]$$

$$= 2.43 \times 10^6 \text{ mm}^4$$

$P_{XY} = -1.62 \times 10^6$ mm^4, as already calculated in Example 1.30.

The MI about principal axes are $I_{1,\,2}$ and are given by

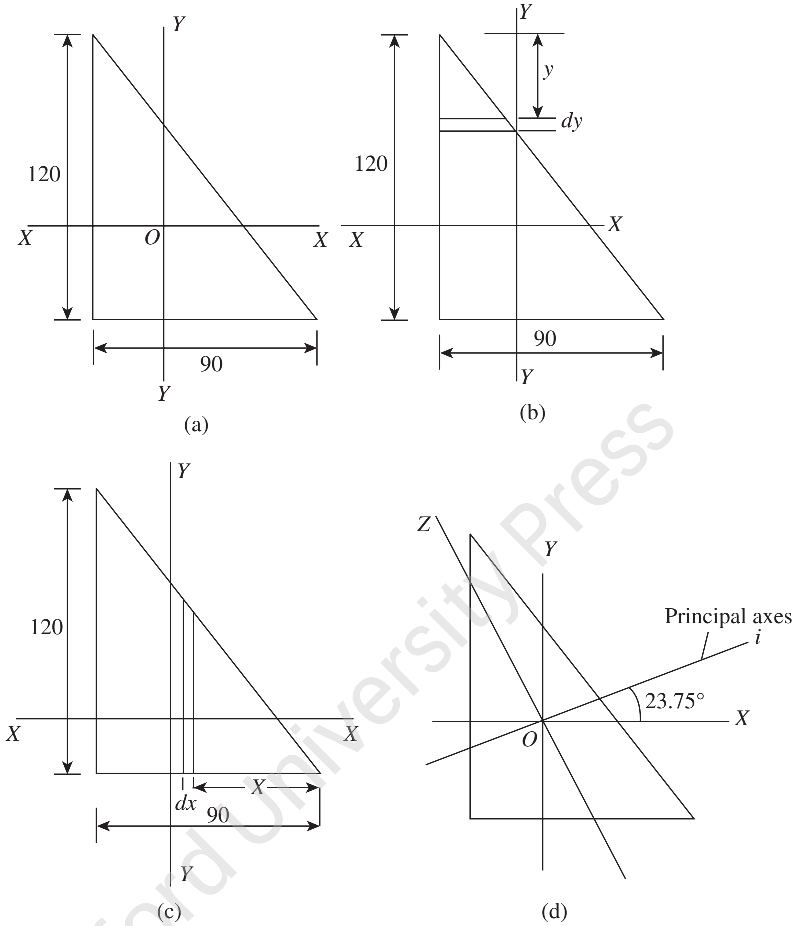


Fig. 1.67

$$I_{1,2} = \frac{I_{XX} + I_{YY}}{2} \pm \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}$$

$$= \frac{4.32\times10^6 + 2.43\times10^6}{2}$$

$$= \pm\sqrt{\left(\frac{4.32\times10^6 - 2.43\times10^6}{2}\right)^2 + (-1.62\times10^6)^2}$$

$$= 5.25 \times 10^6 \text{ mm}^4 \quad \text{or} \quad 1.5 \times 10^6 \text{ mm}^4$$

The inclination of the principal axes is given by

$$\tan 2\alpha = \frac{P_{XY}}{\left(\frac{I_X - I_Y}{2}\right)} = \frac{(-1.62)\times10^6}{\left(\frac{2.7832 - 1.0032}{2}\right)\times10^6} = 1.091$$

$$2\alpha = 59.74° \Rightarrow \alpha = 29.87°$$

The principal axes are shown in Fig. 1.67(d).

Example 1.34 Principal axes and principal MI for an angle section

Calculate the moment of inertia about the principal axes through the centroid of the angle section shown in Fig. 1.68(a).

Solution We first locate the position of the centroid:

$$(120 \times 10 + 70 \times 10)\,\bar{x} = 120 \times 10 \times 5 + 70 \times 10 \times 45$$

$$\bar{x} = 19.73 \text{ mm}$$

$$(120 \times 10 + 70 \times 10)\,\bar{y} = 120 \times 10 \times 60 + 70 \times 10 \times 5$$

$$\bar{y} = 39.74 \text{ mm}$$

I_{XX}, I_{YY}, and P_{XY} can be calculated from Fig. 1.68(b):

$$I_{XX} = \frac{10 \times 120^3}{12} + 1200\,(60 - 39.74)^2 + \frac{70 \times 10^3}{12} + 700 \times (39.74 - 5)^2$$

$$= 2.7832 \times 10^6 \text{ mm}^4$$

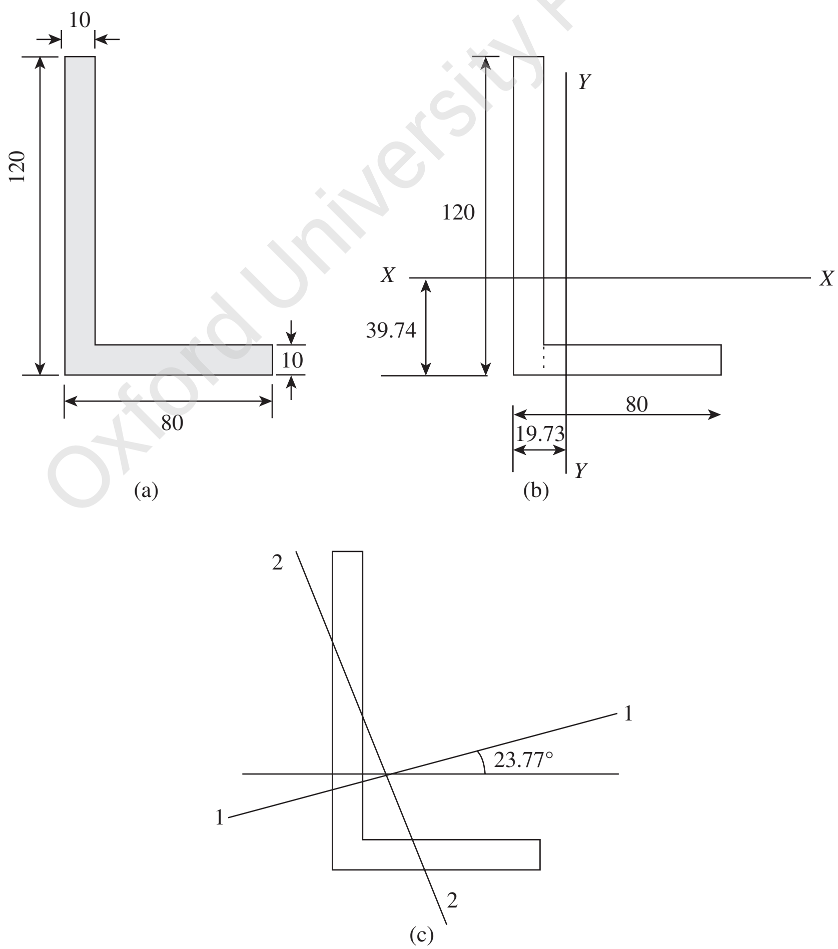


Fig. 1.68

$$I_{YY} = \frac{120 \times 10^3}{12} + 1200 \times 14.73^2 + \frac{10 \times 70^3}{12} + 700 \times 25.27^2$$

$$= 1.0032 \times 10^6 \text{ mm}^4$$

$$P_{XY} = 1200 \times 14.73 \times (-20.26) + 700 \times 34.74 \times (-25.27)$$

$$= -0.9726 \times 10^6 \text{ mm}^4$$

If 1 and 2 are the principal axes through the centroid,

$$I_{1,2} = \frac{I_{XX} + I_{YY}}{2} \pm \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}$$

Therefore,

$$10^6 = \left[\frac{2.7832 + 1.0032}{2} \pm \sqrt{\left(\frac{2.7832 - 1.0032}{2}\right)^2 + (-0.9726)^2}\right]$$

$$= 3.2125 \times 10^6 \text{ mm}^4 \quad \text{or} \quad 0.5759 \times 10^6 \text{ mm}^4$$

The inclination of the principal axes is given by

$$\tan 2\alpha = \frac{-P_{XY}}{\left(\frac{I_X - I_Y}{2}\right)} = \frac{(-0.9726) \times 10^6}{\left(\frac{2.7832 - 1.0032}{2}\right) \times 10^6} = 1.0928$$

$2\alpha = 47.54°$, $\alpha = 23.77°$. The principal axes are shown in Fig. 1.68(c).

1.13 MOHR'S CIRCLE FOR MI

There is an elegant graphical construction to help determine the principal MIs and the orientation of the principal axes. The concept involved is very similar to that of the principal stresses discussed later, in Chapter 8. The graphical construction of Mohr's circle is outlined below. Given I_{XX}, I_{YY}, and P_{XY} about the *X-X* and *Y-Y* axes, we have to find the principal MI and the orientation of the principal axes.

1. Draw a horizontal line which is the *I*-axis and a line perpendicular to it which is the *P*-axis (Fig. 1.69). You have thus an *I-P* coordinate system. The intersection point *O* is the origin.
2. From the origin, mark I_{XX} and I_{YY} to some scale.
3. If *OB* represents I_{YY} and *OA* represents I_{XX}, then $AB = (I_{XX} - I_{YY})$.
4. Bisect *AB* to obtain the point *C*. $OC = (I_{XX} + I_{YY})/2$.
5. Draw perpendicular lines at *B* and *A* and mark *D* and *E* such that $BD = P_{XY} = AE$. The sign of P_{XY} is associated with I_{XX} and the opposite sign with I_{YY}.
6. With *CD* (or *CE*) as the radius and the centre at *C*, draw a circle.
7. Since $AC = (I_{XX} + I_{YY})/2$ and $AE = P_{XY}$, $CE = \{[(I_{XX} - I_{YY})/2]^2 + P^2_{XY}\}^{1/2}$, which is the radius of the circle. This is known as Mohr's circle.
8. Let the circle intersect the *I*-axis at points 1 and 2.
9. The coordinates of points 1 and 2 give the principal moment of inertia. Note that $P_{XY} = 0$ at points 1 and 2. This statement can be easily proved:

$$O2 = OC - C2$$

$$= \frac{I_{XX} + I_{YY}}{2} - \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}$$

as $OC = (I_{XX} + I_{YY})/2$ and $C2$ is the radius of Mohr's circle

$$= \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}$$

Similarly,

$O1 = OC + C1$, $C1$ being the radius of the circle

$$= \frac{I_{XX} + I_{YY}}{2} + \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}$$

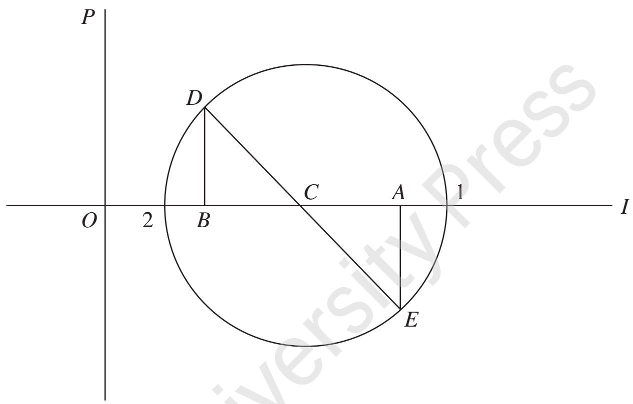


Fig. 1.69

10. From the triangle CAE, $\tan 2\theta = \dfrac{-P_{XY}}{[(I_{XX} - I_{YY})/2]}$.
11. It can also be proved that the coordinates of any point in the circle represent the I and P about some axis. The diametrically opposite point gives I and P about an axis perpendicular to it. We will discuss these concepts in Chapter 9 in greater detail. With reference to principal stresses, which are applicable to principal moments of inertia as well.

The following example illustrates the procedure.

Example 1.35 Principal axes and principal MI for an angle section using Mohr's circle

For the angle section of Example 1.32, find the principal axes and inertias graphically using Mohr's circle.

Solution The scale chosen is 1 cm = 0.25×10^6 mm^4.

OI and OP are perpendicular axes.

$I_{XX} = 2.7832 \times 10^6$ mm^4, $I_{YY} = 1.0032 \times 10^6$ mm^4, $P_{XY} = -0.9726 \times 10^6$ mm^4.

In Mohr's circle, $OA = I_{XX}$, $OB = I_{YY}$, $AE = BD = P_{XY}$.

Join BD intersecting the I-axis at c. With C as the centre and CD as the radius, draw a circle intersecting the I-axis at points 1 and 2.

Points 1 and 2 being on the I-axis, $P_{XY} = 0$ about these axes.

Fig. 1.70 shows Mohr's circle drawn using the given values. $O1$ and $O2$ give the principal MI, and $\angle ACE = 2\theta$.

$O1 = 12.85 \times 0.25 \times 10^6 = 3.2125 \times 10^6$ mm^4

$O2 = 2.3 \times 0.25 \times 10^6 = 0.575 \times 10^6 \text{ mm}^4$

$\angle ACE = 47°, \quad 2\theta = 47°, \quad \theta = 23.5°$

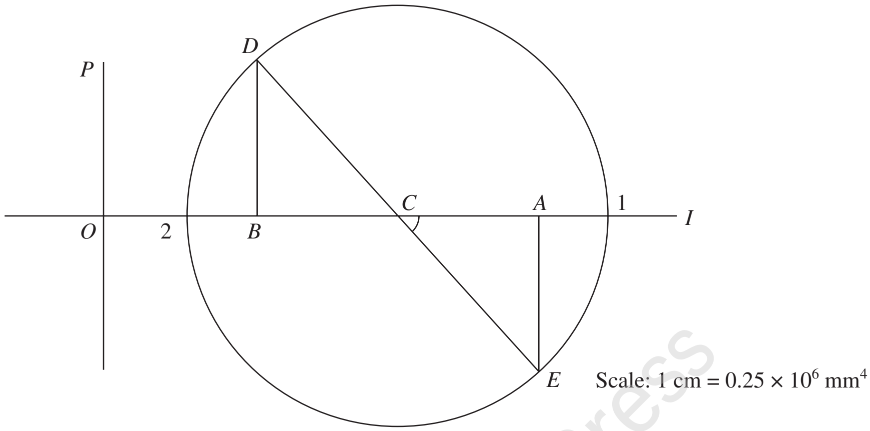


Fig. 1.70

1.14 GRAPHICAL CONSTRUCTION TO FIND MOMENTS OF INERTIA

A simple graphical procedure is available to find the MIs of irregular areas.

Consider the area shown in Fig. 1.71, whose MI is to be found about axis *X*-*X*. Draw a line *X*′-*X*′ parallel to *X*-*X* on the other side as shown in the figure.

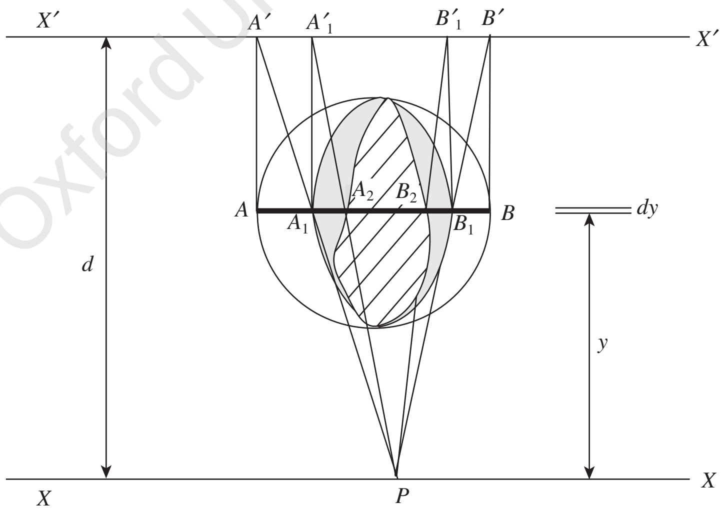


Fig. 1.71

Take a narrow strip *AB* of thickness *dy* in the area and project it on x′-x′ as *A*′ *B*′. Take a suitable point *P* on *X*-*X* and draw *PA*′ and *PB*′ such that they intersect *AB* at A_1 and B_1. The area of strip A_1B_1 can be proved to be the first moment of area of strip *AB* about *P*. By repeating the procedure with other strips, one can obtain the shaded figure. The area of this figure multiplied by *d* is equal to the moment of the area of the given figure about *P*.

Consider the strip A_1B_1 and repeat the procedure. Project A_1B_1 on X'-X' as $A_1'B_1'$ and draw PA_1' and PB_1'. Repeat the procedure with other strips of the shaded figure and obtain the hatched figure. The area of the hatched figure multiplied by d is the MI of the area about axis AB. This can be proved as follows.

$$\frac{A'B'}{d} = \frac{A_1B_1}{y}$$

because $A'B'P$ and A_1B_1P are similar triangles.

$$A'B'y = A_1B_1d$$

Since $A'B' = AB$,

$$A_1B_1d = ABy$$

Multiplying by dy,

$$A_1B_1ddy = ABydy = (ABdy)y$$

where $AB\ dy$ is the area of the strip, which multiplied by y gives the moment of this area about P. A_1B_1dy is the area of the shaded strip, and this multiplied by d is thus equal to the first moment of area of the strip $AB\ dy$. This is true for all the elementary strips. The area of the shaded figure multiplied by d, the distance between X-X and X'-X', therefore, is equal to the first moment of area about P. The hatched figure is obtained by a similar procedure from the shaded figure. That is, the hatched figure is the first moment of the shaded figure about P. If A is the given area, A_1 is the area of the shaded figure, and A_2 is the area of the hatched figure, then

$$A\,\bar{y} = A_1d \quad \text{as proved earlier}$$

$$\text{MI} = A\,\bar{y}\,\bar{y}\ (A_1\,\bar{y})\,\bar{y}, \quad A_1\,\bar{y} = A_2d \quad \text{(on similar arguments)}$$

$$\text{MI} = A_2d \times d = A_2\,d^2$$

The moment of inertia of the given area is equal to area A_2 multiplied by the square of distance d between X-X and X'-X'.

1.15 STRUCTURAL ENGINEERING

There are different types of structures built by man—buildings, bridges, culverts, water tanks, storage bins, roads, transmission towers, machines, etc. Such structures are built up of a number of structural elements joined together suitably.

In the case of any structure, two types of designs are involved—functional or architectural and structural. Functional or architectural design deals with aspects other than the strengths of the structure, like aesthetics, utility, orientation, general layout, etc. Once this important aspect of design is taken care of, structural designers take over. Their work involves analysis of the structure and its elements to find the forces and moments that they have to withstand and then design the dimensions of the elements and their interconnections. The former is the realm of structural analysis while the latter forms what is called structural design.

Structures are designed to withstand loads, i.e., forces and moments due to different causes. While some structures, like aircraft structures, machine foundations, etc. necessarily have to be designed for forces due to motion, many are designed considering them to be in equilibrium, or at rest. A body in uniform motion is also governed by the same laws as for a body at rest. In this book, we will deal with structures which are in equilibrium.

1.16 STRUCTURAL ELEMENTS AND STRUCTURAL BEHAVIOUR

When subjected to loads, structures may deform due to the straining action of the loads. There are basically three types of straining action—tensile, which tends to elongate the fibres of the element; compressive, which tends to shorten the fibres; and shearing, which is an action tangential to the cross section of the fibre. These are shown in Fig. 1.72. Note that while tensile and compressive strains are along the length of the fibre, shear strain is along the cross section of the fibre. These fibres act like springs and resist such straining action by developing stresses. The resultant of such resisting action is known as a stress resultant. When an equilibrium is reached between the actions and stress resultants, the element stays in equilibrium. There is work done by the applied forces in straining the element and this is stored as elastic strain energy in the element. Under normal conditions, when the applied forces are removed, the strains disappear and the structure comes back to its original dimensions. These concepts are explained in detail in later chapters.

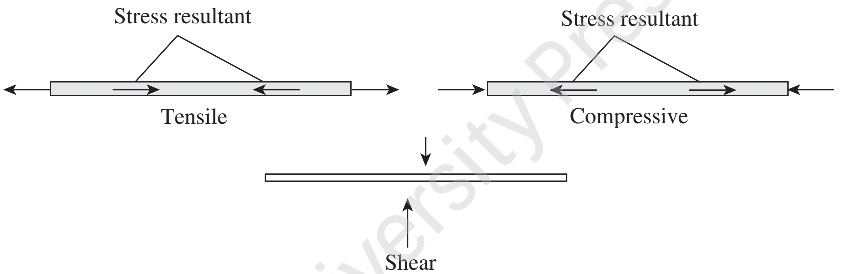


Fig. 1.72

There are different types of structures, consisting of different kinds of elements which exhibit different structural actions. Let us discuss them briefly.

Tension member A tension member can be represented as shown in Fig. 1.73(a). The action is equivalent to that of two forces tending to stretch the element, and is normal to the cross section, which may have any shape. The member increases in length due to the straining action, and develops stress resultants to oppose the applied forces as shown.

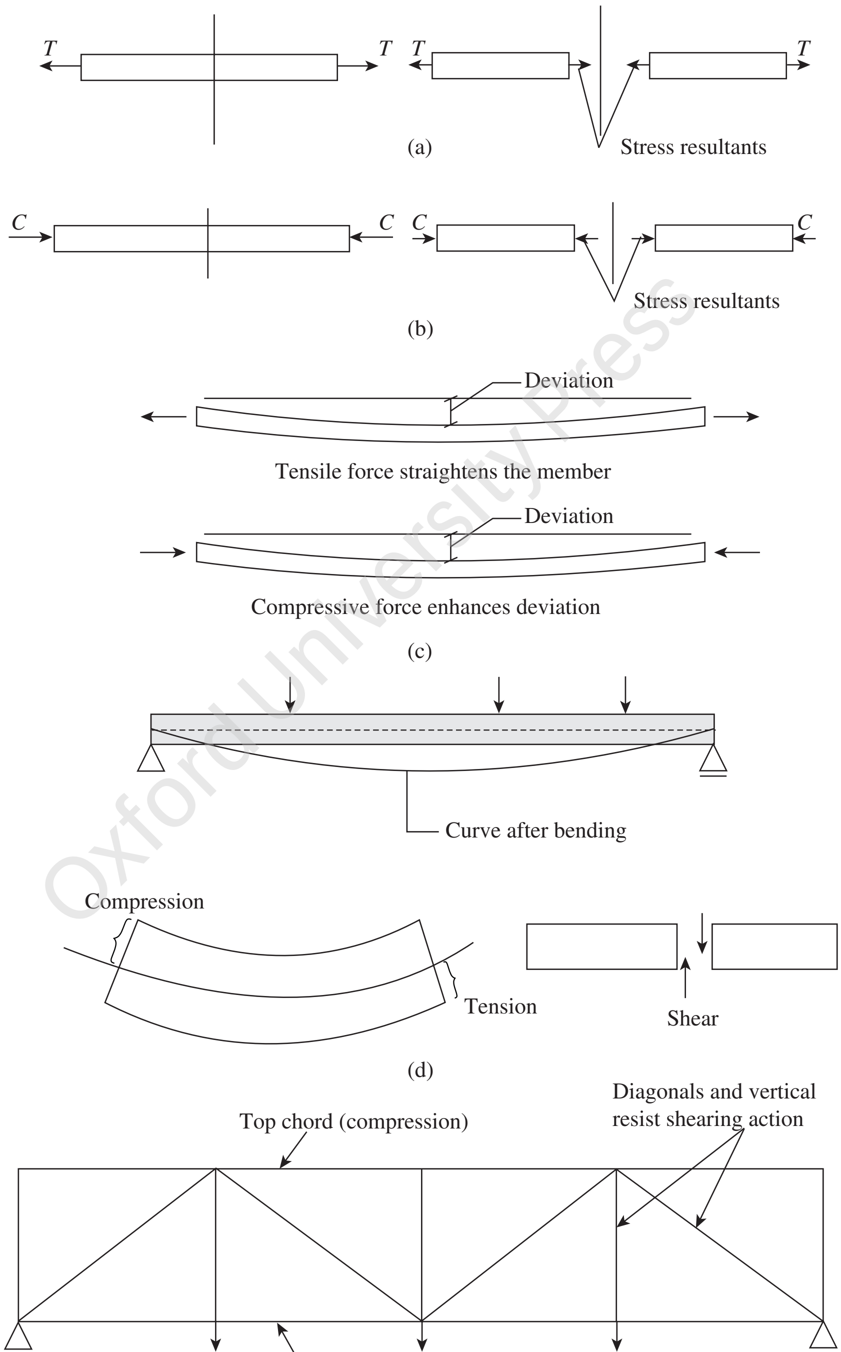
Compression member A compression member can be represented as shown in Fig. 1.73(b). The two actions tend to shorten the member, and the stress resultants are directed as shown.

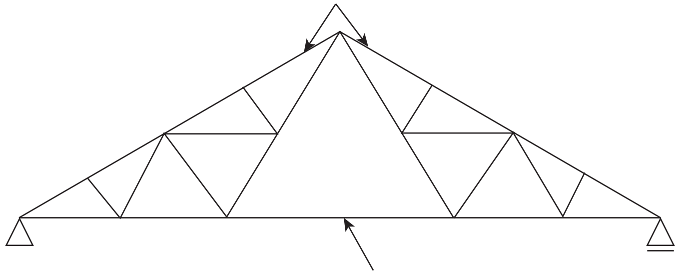
A compression member has such a behaviour only when it is small in length. The behaviour changes as the length increases. The applied actions tend to shorten the member and also bend it. The bending action becomes more prominent for long members and this phenomenon is known as buckling.

Another difference between tension and compression members may be noted. While tension tends to reduce the defect in the case of any slight deviation in the straightness of the member, as can happen in fabrication, compressive forces enhance the defect [Fig. 1.73(c)].

Beams Beams are very common structural elements used to span distances. They carry loads predominantly transverse to their longitudinal axis [Fig. 1.73(d)]. Due to the straining action of the applied loads, the beam tends to bend, i.e. take up the curved shape shown. Beams are subjected to all the three straining actions described above. The top part of the beam is subjected to compression, the bottom part to tension, and there is also shearing action parallel to the cross section.

Trusses Trusses are used to span large distances. A truss may have sloping members on top as shown in Fig. 1.73(e). In the ideal condition, the members forming the truss framework are tension and compression members with loads acting at the joints only. On the whole, they





(e) Truss with sloping members

Fig. 1.73

behave like beams—in the case of a parallel chorded truss, the top, chord members are in compression, the bottom chord members are in tension, and the diagonal or vertical members resist the shearing action. In terms of structural action, one can consider that the truss acts like a beam from which certain parts have been cut out, but without the bending effect. Also, in the case of a truss, the shearing action is resisted by tension and compression in the diagonal members.

Arches Arches are again elements used to span large distances. They basically carry load to supports by developing compressive stresses in them. In practice, some bending and shear are also developed. The supports for an arch need to be strong because the arch transfers the load at an inclination to the supports. Arches may be three-hinged, two-hinged, or fixed (Fig. 1.74). The load is transferred to the arch through spandrels or hangers to make them uniform.

If the arch has a funicular shape with reference to the load, the arch is under pure compression. For example, for a uniformly distributed load, the funicular shape is a parabola.

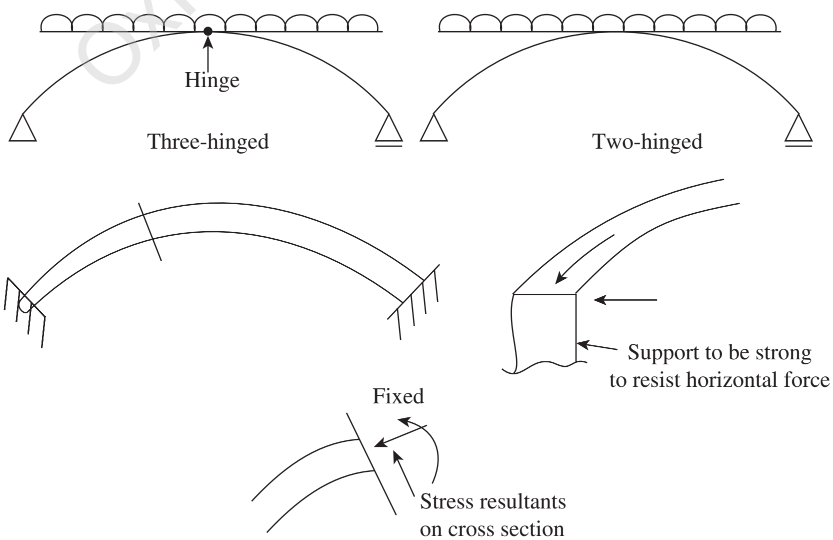


Fig. 1.74

Plates A plate is a two-dimensional structure, and is generally subjected to loads perpendicular to its plane and supported along the edges or in many other ways. The behaviour of the plate element is very complex but can be simplified by assuming beam action of strips of the plate in two perpendicular directions (Fig. 1.75). The deflected shape is that of a saucer, with curvature in two directions. This is a very common element used as roof and floor slabs, bridge and culvert decks, etc.

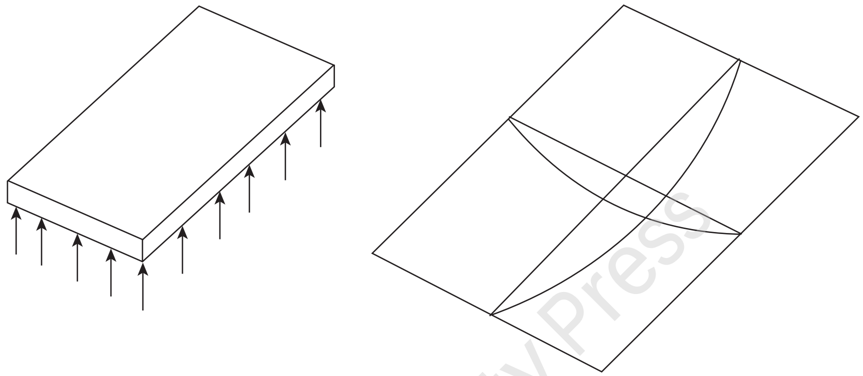


Fig. 1.75 Plate action

Cable A cable, as mentioned earlier, is effective only when stretched. It finds use in bridge structures as shown in Fig. 1.76, and is basically a tension member. A cable supported at ends hangs in a shape known as 'catenary' due to its own weight. Subjected to loads, the flexible cable takes up a shape dictated by the loads acting on it.

Rigid frames A rigid frame is a framework of elements, monolithically cast or otherwise rigidly jointed (unlike in a truss where the members are assumed to be pin-jointed). Such a framework is very common in multi-storeyed, multibay buildings. The top members may be inclined as a gabled portal (Fig. 1.77). The vertical members predominantly act as columns or compression members but are also subjected to beam action. The horizontal members are beams but may also carry axial forces. The behaviour of such a framework is very complex due to the connecting members and plates both ways. At any interior joint, there are beams from all four directions and vertical members above and below. The slabs (plates) connected to the beams also affect the behaviour of these members.

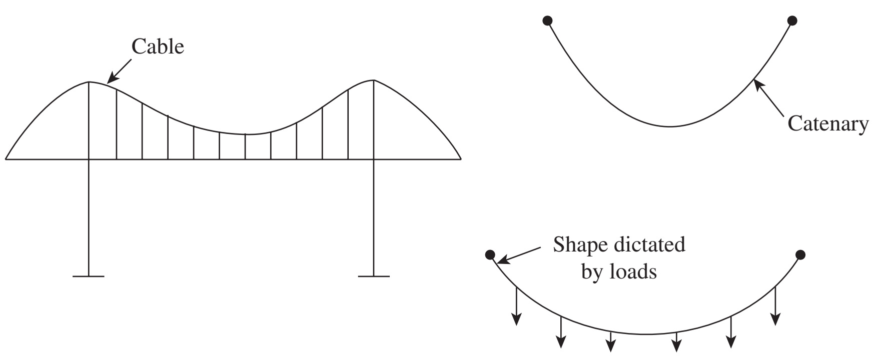


Fig. 1.76

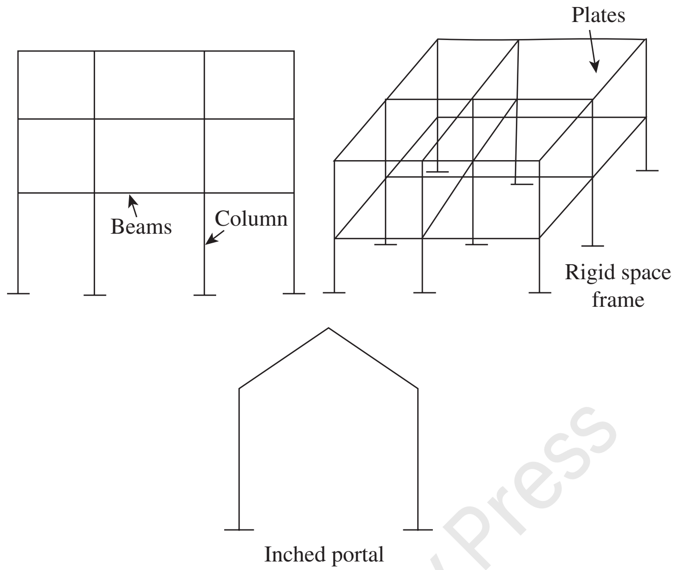


Fig. 1.77

Grids A grid structure is shown in Fig. 1.78. The interconnected elements both ways behave like beams but the structure is efficient in distributing the effect of loads on an element both ways. They have become common in roof structures. The grid structure can also be diagonally made resulting in a skew grid. In addition to bending, the elements are also subjected to a twisting action.

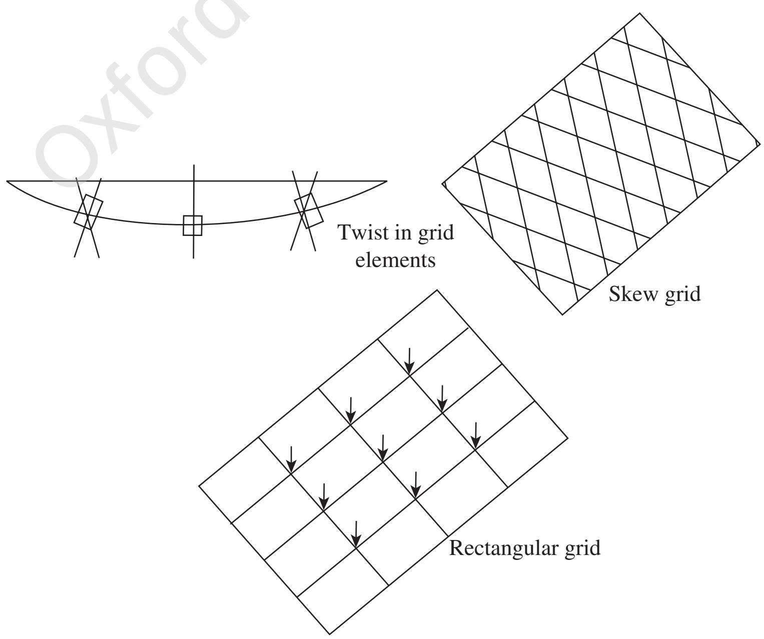


Fig. 1.78

Shell structures Shell structures have become very popular in recent times with a better understanding of their behaviour even though massive shells have been made since the early stages of civilization. There are innumerable forms of shells. Some of these are shown in Fig. 1.79.

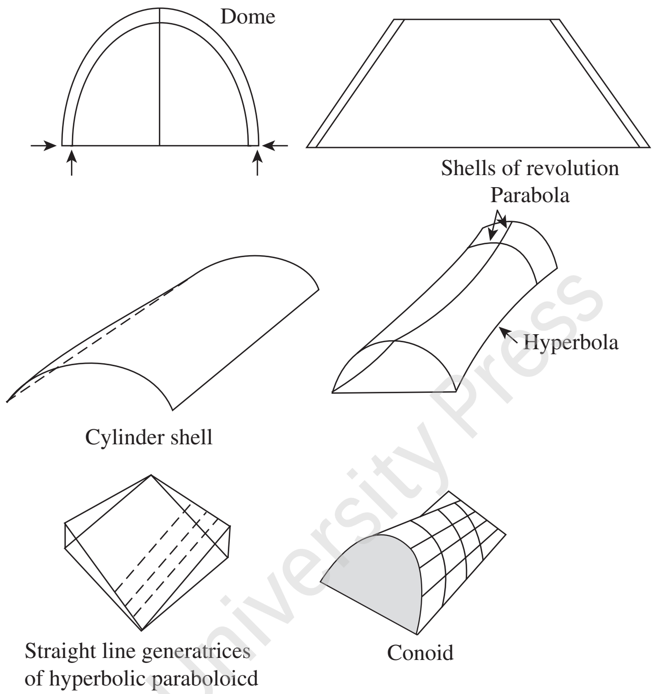


Fig. 1.79

Shells of revolution are those generated by the rotation of an arc about a line. Examples are different types of domes and conics. They are essentially thin elements, subjected to meridional and hoop stresses.

Translational shells are generated by the movement of a line over end arcs, as in a cylindrical shell or one curve over another perpendicular curve, as in a hyperbolic paraboloid. A hyperbolic paraboloid can also have two sets of straight lines lying on its surface. Many such surfaces, called ruled surfaces, can be generated by moving a straight line along two separate curves, as in a conoid.

More complex shell shapes can be obtained by intersecting surfaces, and have been used in structures for their aesthetic value. Shells are very thin elements and their thickness is governed by the practicality of fabrication rather than structural needs.

1.17 STRUCTURAL DESIGN: STRENGTH, STIFFNESS, AND STABILITY

Structural analysis provides with forces and moments that the elements of the structure have to withstand. Using these forces and moments and the material properties, we derive the dimensions of the structural elements. This is called *structural design*. There are three considerations in the design of structures or machine components.

Strength design is done to ensure that the stresses at any point in the element do not exceed the permissible value for the material. The structural element may be made of a single material like steel and timber. The material properties may be different in tension and compression as in timber. The element may also be made of two or more materials as in a composite element. Strength design ensures that in all cases, the stress in the element is within the appropriate permissible values. This is discussed in detail in Chapter 7 on Deformations in Beams.

Stiffness design is another aspect of design. Stiffness relates to the deformation of the member. In general, stiffness is mathematically defined as the force/moment required to cause unit deformation in the member. You have thus different stiffness for a member depending upon the forces and moments that it has to carry. You have thus axial stiffness, flexural (bending) stiffness, torsional stiffness, etc.

A member may be strong to carry the forces and moments but may have unacceptable deformations. Consider a steel rod, 6 m long, being carried by holding it at the ends [Fig. 1.77(b)]. The rod will not break if it is carried like that but will sag considerably in the middle. The rod is strong but weak in stiffness in this position.

Stability is another consideration in structural design. A dam, for example, may be strong enough to carry the loads and stiff enough because of its size [Fig. 1.80 (c)]. But because of the horizontal forces, it may have rigid body rotation leading to overturning. This is discussed in Chapter 6. Long columns subjected to axial forces also have stability considerations in the design (Chapter 11).

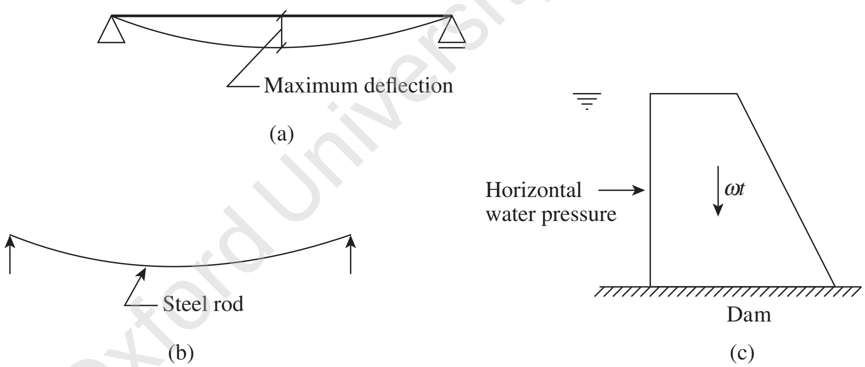


Fig. 1.80

1.18 SYMBOLS AND UNITS

A number of symbols are used in this book. They are explained wherever they are first encountered. A collection of such symbols is given at the beginning of the book. SI units will be used throughout in this work.

Summary

Structures, machine components, and other similar elements are designed to withstand forces and moments. Such designs, which belong to the realm of structural engineering or design theories, are based on the basic principles of statics and dynamics. Statics is the branch of mechanics dealing with forces and moments when the body is in equilibrium. The present discussion is limited to structures and components in equilibrium.

Force systems can be concurrent, parallel or non-concurrent. Resultants of force vectors are obtained by adding them according to the rules of vector algebra, which are different from those of scalar algebra. Two vectors are added according to the parallelogram law or triangle law. Force systems can also be coplanar or space systems. In all cases, forces and moments have to be added as vectors.

The resultant of two forces F_1 and F_2 is given by $R = (F_1^2 + F_2^2 - 2F_1F_2 \cos\theta)^{1/2}$, where θ is the angle between the vectors. More than two forces can be added by repeated application of this principle. Forces can also be added by resolving them into rectangular components in two mutually perpendicular directions. The sum of such components, ΣFx and ΣFy, can then be combined into a single force $R = \left[(\Sigma Fx)^2 + (\Sigma Fy)^2\right]$.

Graphical methods use the force polygon and funicular polygon to find the resultant force and its location.

Body constraints are supports provided to a body to give forces and moments needed to keep the body in equilibrium. Free body diagrams are diagrams showing the body acted upon by the applied forces and moments, and the reactive forces and moments given by the body constraints. Once such a diagram is drawn, the reactive forces and moments can be determined by the equilibrium conditions, such as $\Sigma Fx = 0$, $\Sigma Fy = 0$, and $\Sigma M = 0$. Such conditions are used to calculate the reactive forces and moments, and analyse structures and machine components for design.

The concepts of centre of gravity and centroid of bodies and areas is also important in structural analysis. These also help us to locate the resultant of distributed forces.

The moment of inertia (MI) or second moment of area is an important property of a section. It is calculated using the integral $\int y^2 dA$. The MIs of standard sections are listed in tables. The second moment of area is always a positive quantity.

The parallel axis theorem states that if I_{GG} is the second moment of area about an axis through the centroid, then the second moment of area about a parallel axis *A-A* is given by $I_{AA} = I_{GG} + Ad^2$, where d is the distance between the centroidal axis and the axis *A-A*.

The polar moment of inertia exists about an axis perpendicular to the area. $I_{ZZ} = I_{XX} + I_{YY}$, where *X-X* and *Y-Y* are two mutually perpendicular axes in the plane of the area and *Z-Z* is an axis perpendicular to the plane and passing through the intersection of the axes *X-X* and *Y-Y*.

The section modulus of a section is the moment of inertia divided by the distances to the extreme fibres of the section. Section modulus is useful in the design of beams.

The product of inertia $P_{XY} = \int xy\ dA$. It can be positive, negative, or zero. $P_{XY} = 0$ if either of the axes *X-X* or *Y-Y* is an axis of symmetry.

If *X-X* and *Y-Y* are the axes through the centroid of the area and *X′-X′* and *Y′-Y′* are another set of parallel axes in the plane of the area, then $P_{X'Y'} = P_{XY} + A\ \bar{x}\bar{y}$ where $\bar{x}, \bar{y}$ are the coordinates of the centroid of the area with respect to axes *X′*, *Y′* and *A* is the area. This formula is used for calculating the product of inertia (PI) through the axes translated parallel to the centroidal axes.

If the axes are rotated by an angle θ, then the MI and PI about the rotated axes, *U-U* and *V-V*, can be found from

$$I_{UU} = \frac{I_{XX} + I_{YY}}{2} + \frac{I_{XX} - I_{YY}}{2} \cos 2\theta + P_{XY} \sin 2\theta$$

$$I_{VV} = \frac{I_{XX} + I_{YY}}{2} + \frac{I_{XX} - I_{YY}}{2} \cos 2\theta - P_{XY} \sin 2\theta$$

$$P_{UV} = \frac{I_{XX} - I_{YY}}{2} \sin 2\theta + P_{XY} \cos 2\theta$$

The principal axes of a section are those axes about which I_{UU} and I_{VV} are a maximum or minimum.

$$I_{(\max/\min)} = \frac{I_{XX} + I_{YY}}{2} \pm \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + P_{XY}^2}$$

The angle θ for such a case is given by

$$\tan 2\theta = \frac{P_{XY}}{\left(\frac{I_{XX} - I_{YY}}{2}\right)}$$

The product of inertia about the principal axis is zero. The axis of symmetry of an area is the principal axis. The MI of an area can also be found by the graphical method.

EXERCISES

Multiple Choice Questions

1. When a parallelogram is drawn to represent two concurrent force vectors A and B, then one diagonal represents the resultant. The other diagonal represents

   (a) $\frac{|A+B|}{2}$ (b) $\frac{|A-B|}{2}$

   (c) $\frac{|A+B|}{4}$ (d) $|A - B|$

2. Two parallel forces, 10 kN↑ and 20 kN↓ are 1 m apart and act on a body. The resultant of these two forces will act
   (a) at a point within the lines of action of these forces
   (b) at a point outside, near the 20 kN force
   (c) at a point outside, near the 10 kN force
   (d) at the point where the 20 kN force acts

3. When two forces keep a body in equilibrium, the two forces must be
   (a) equal
   (b) equal and collinear
   (c) equal, collinear and opposite
   (d) concurrent

4. The moment of the force of 1 N (refer to Fig. 1.81) about axes 1-1, 2-2, 3-3 respectively are (in Nm)

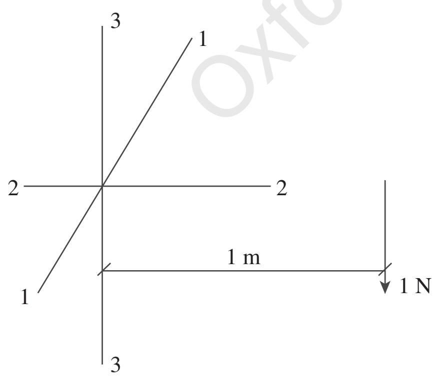


Fig. 1.81

   (a) 1, 1, 1 (b) 1, 1, 0 (c) 1, 0, 0 (d) 0, 0, 0

5. Varignon's theorem or the principle of moments states that
   (a) $\Sigma M = 0$ for a coplanar force system in equilibrium
   (b) $\Sigma M = 0$ for a parallel force system
   (c) $\Sigma M \neq 0$ about any point at which two forces intersect
   (d) ΣM of individual forces about a point is equal to the moment of their resultant about the same point

6. When the conditions of equilibrium for a general coplanar system is taken as $\Sigma M_1 = 0$, $\Sigma M_2 = 0$ and $\Sigma M_3 = 0$ about points 1, 2, and 3, then
   (a) the points 1, 2, 3 lie on the lines of action of the forces
   (b) the points 1, 2,3 should not lie on the lines of action of forces
   (c) the points 1, 2, 3 should not be collinear
   (d) the points 1, 2, 3 should lie on the arc of a circle.

7. For the shaded area of Fig. 1.82, the most probable position of centroid is

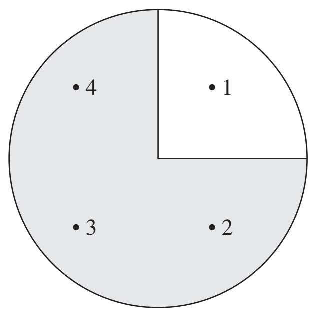


Fig. 1.82

   (a) 1 (b) 2 (c) 3 (d) 4

8. In the figure shown in Fig. 1.83, the most probable location of centroid is

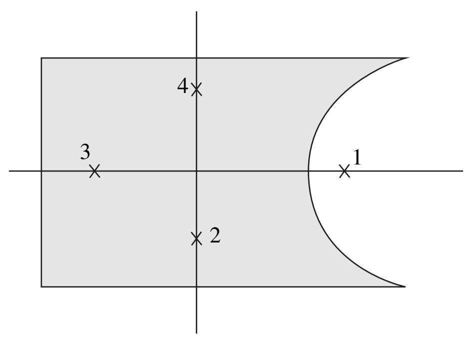


Fig. 1.83

   (a) 1 (b) 2 (c) 3 (d) 4

9. The resultant of the distributed load shown in Fig. 1.84 is

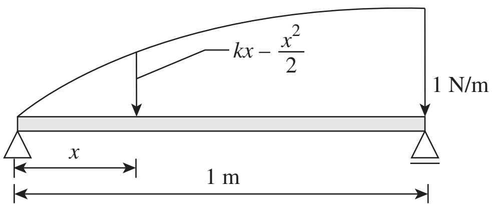


Fig. 1.84

(a) 1/12 (b) 5/12 (c) 7/12 (d) 9/12

10. The beam carries a distributed load as shown in Fig. 1.85 The reactive force at *A* is

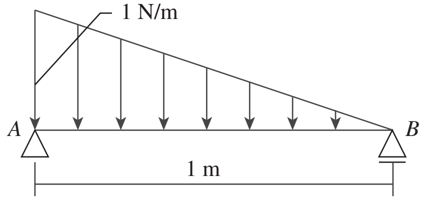


Fig. 1.85

(a) 1/8 N (b) 1/6 N (c) 1/4 N (d) 1/3 N

11. The reaction at *A* in the case shown in Fig. 1.86 is

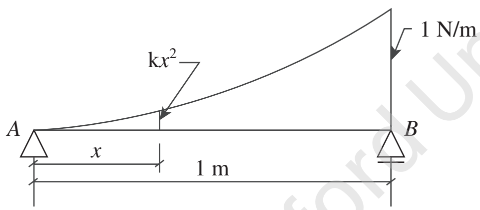


Fig. 1.86

(a) 1/3 N (b) 1/4 N (c) 1/6 N (d) 1/12 N

12. In Fig. 1.87, the tension in the cable is (in N)

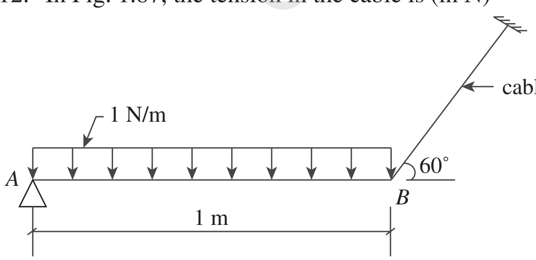


Fig. 1.87

(a) 1/4 (b) 1/2 (c) $1/\sqrt{3}$ (d) 1

13. The moment of inertia of a hollow circular section of inner radius *r* and thickness $0.1r$ will be
(a) $\pi[r^4 - (0.1r)^4]/4$ (b) $\pi[(1.1r)^4 - r^4]/4$
(c) $\pi(1.1r^4 - r^4)/64$ (d) $\pi(r^4 - 0.9\,r^4)/4$

14. The second moment of area of an equilateral triangle section of side *a* about an axis through its base will be
(a) $0.054\,a^4$ (b) $0.083\,a^4$
(c) $0.433\,a^4$ (d) a^4

15. The true statement from the following is
(a) second moment area is always positive
(b) Product of inertia is always positive
(c) If second moment of area is positive, product of inertia will also be positive
(d) Product of inertia is always negative.

16. In an area in the form of a triangle, two mutually perpendicular axes are considered through its centroid, one of the axes being parallel to the base. The true statement about the product of inertia (PI) of the area about such axes from the following is
(a) PI is zero for an equilateral triangle
(b) PI is zero for a right-angled triangle
(c) PI is zero for all triangles
(d) PI is not zero for any triangle

17. If an equilateral triangle of side '*a*' and a square of side '*b*' have the same second moment of area about an axis through their centroid parallel to the side, the ratio *a/b* is
(a) 0.732 (b) 1 (c) 1.466 (d) 2.930

18. For a circular area of radius *r*, the radius of gyration with respect to an axis tangent to the area is
(a) *r*/2 (b) *r* (c) 1.12 *r* (d) 1.5 *r*

19. The radii of gyration of i) a circular area of radius *r* about its diameter and ii) a square of side a about its centroidal axis parallel to the side are equal. The ratio *r/a* is
(a) 0.289 (b) 0.577
(c) 1 (d) 1.154

20. The radius of gyration of a hollow circular area of inner radius *r* and thickness $0.1r$ with respect to an axis through its centre is
(a) 0.371 *r* (b) 0.743 *r*
(c) *r* (d) 1.104 *r*

21. The unit of section modulus is
(a) mm (b) mm^2
(c) mm^3 (d) mm^4

22. The section modulii of an equilateral triangular section, about an axis through the centroid and parallel to the base is
(a) $a^3/4$, $a^3/8$ (b) $a^3/8$, $a^3/16$
(c) $a^3/16$, $a^3/32$ (d) $a^3/32$, $a^3/64$

23. A square section of side 'a' is (i) kept with the base horizontal and (ii) with the diagonal horizontal. The ratio of section modulus of position (i) to that in (ii) is
    (a) 0.707 (b) 1.414 (c) 2 (d) 2.828

24. An area in the shape of a regular hexagon of side 'a' is kept with i) one side horizontal and ii) with the same side vertical. The ratio of section modulus of position (i) to that of position (ii) is
    (a) 1.58 (b) 3.15 (c) 4.74 (d) 6.32

Review Questions

1. A force is to be resolved into two rectangular components. Is there a unique solution? Explain your answer. When does the solution become unique?
2. Can the component of a force be larger in magnitude than the force? Explain your answer.
3. A body is acted upon by three forces F_1, F_2, and F_3, which act along the sides of a triangle and are proportional to these sides. Is the body in equilibrium? If it is not, under what conditions will it be?
4. A body is acted by four forces F_1, F_2, F_3, and F_4, which act along the sides of a rectangle, and whose magnitudes are proportional to the respective sides. Is the body in equilibrium? If not, under what conditions the body will be in equilibrium?
5. Explain why it is not necessary to use the principle of moments to find the resultant of a concurrent force system.
6. If $\Sigma F = 0$, for a parallel force system, what can you say about the resultant of this system.
7. Can the funicular polygon close if the force polygon does not close? Explain your answer.
8. If the first and last lines of a funicular polygon are parallel, but not collinear, explain how you would calculate the resultant, with a diagram.
9. If, for a general coplanar force system, $\Sigma M_1 = 0$, $\Sigma M_2 = 0$, and $\Sigma M_3 = 0$, about three moment centres 1, 2, and 3, under what conditions will the force system be in equilibrium? Give reasons for your answer.
10. For the load distribution shown in Fig. 1.88, will the resultant of the load be at 1, 2, or 3?

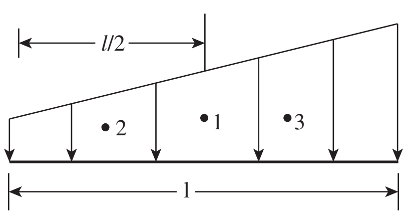


Fig. 1.88

11. If a beam is loaded as shown in Fig. 1.89, what will be the reaction at A and B?

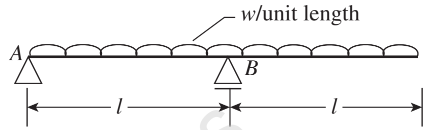


Fig. 1.89

12. Can the MIs of an area about any axis in its plane be zero? Explain your answer.
13. Of the two equal triangles shown in Fig. 1.90, which has a greater MI about the y-axis? Are their MIs about the x-axis equal

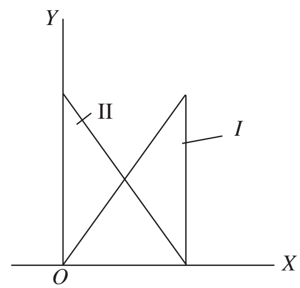


Fig. 1.90

14. The strength of a beam section is directly proportional to the MI of the section about its centroidal, horizontal axis (Fig. 1.91). Which of the two sections, of equal area, is preferable as a beam section? Why

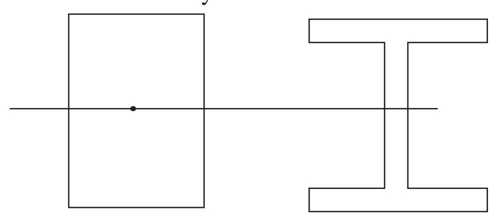


Fig. 1.91

15. Can the radius of gyration of any area be zero?

16. For which of the sections shown in Fig. 1.92, the product of inertia is not zero about the *X-X* and *Y-Y* axes

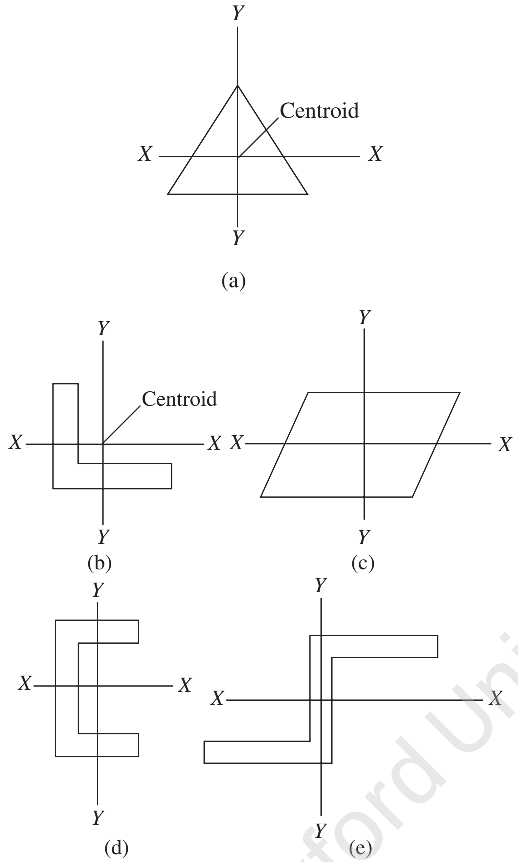


Fig. 1.92

17. In each case shown in Fig. 1.93, state, without calculation, whether the product of inertia is zero, negative, or positive

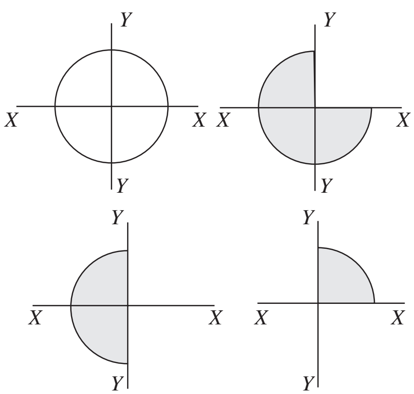


Fig. 1.93

18. In the shapes shown in Fig. 1.94, what are the MIs about the *X-X* axis

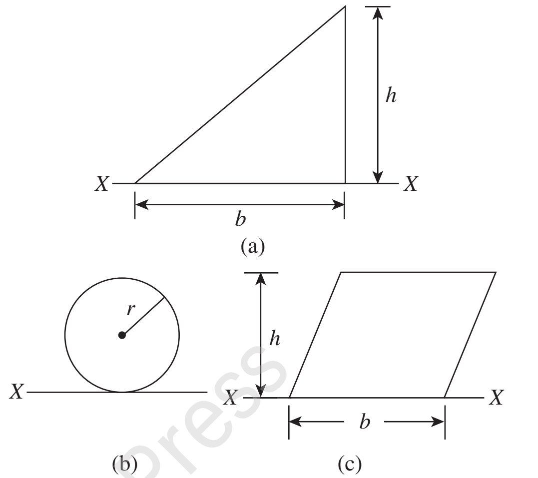


Fig. 1.94

19. In the cases shown in Fig. 1.95, about which axis the MI will be more—*X-X* or *Y-Y*? Why

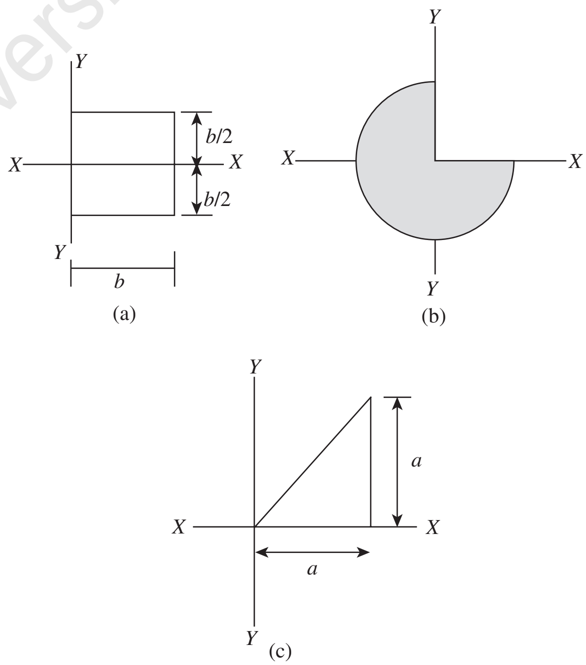


Fig. 1.95

20. Product of inertia can be negative, zero, or positive. Illustrate each of these with examples.
21. Draw sketches of at least two sections each where (i) section modulus is the same and (ii) section moduli are different, with respect to top and bottom fibres.

Problems

1. A force of 200 N is resolved into rectangular components. Find the magnitude and direction of the components if their magnitudes are in the ratio 1 : 2.
2. Resolve the force of 500 N (Fig. 1.96) into parallel components through (i) *A* and *B* and (ii) *A* and *C*.

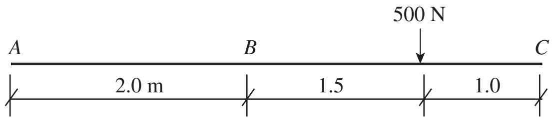


Fig. 1.96

3. A force is resolved into three components. If $\overrightarrow{DB}$ = 100 N, find the magnitudes of $\overrightarrow{AC}$ and $\overrightarrow{CD}$ for the two cases shown in Fig. 1.97. In Fig. 1.97 (a), AB ∥ CD.

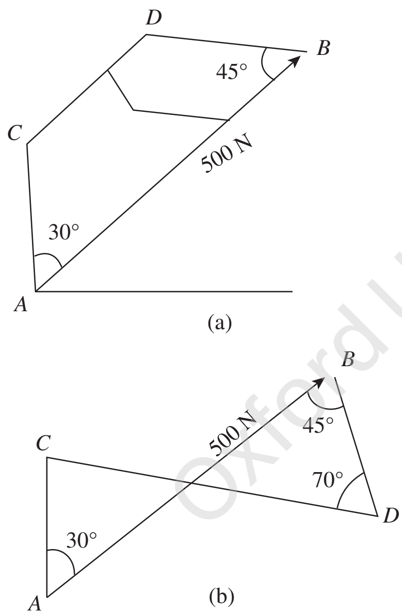


Fig. 1.97

4. A load of 1000 N is supported by two cables as shown in Fig. 1.98. Find the tension in these cables.

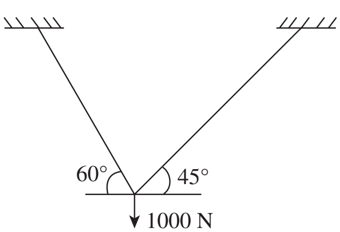


Fig. 1.98

5. Find the resultant of the parallel force system shown in Fig. 1.99, analytically and graphically.

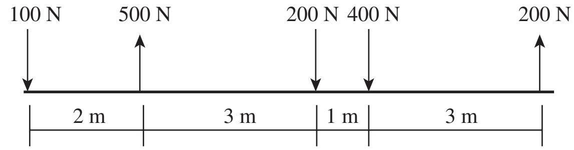


Fig. 1.99

6. Find the resultant of the parallel force system shown in Fig. 1.100, graphically.

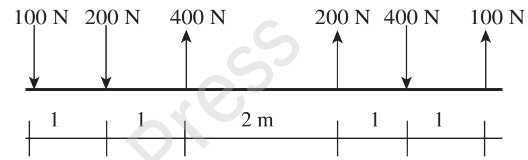


Fig. 1.100

7. Find the resultant of the three forces shown in Fig. 1.101.

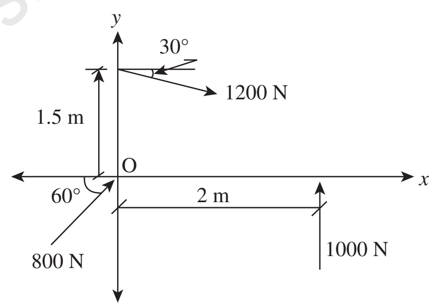
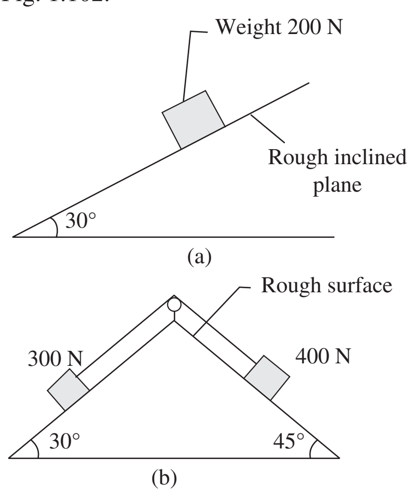
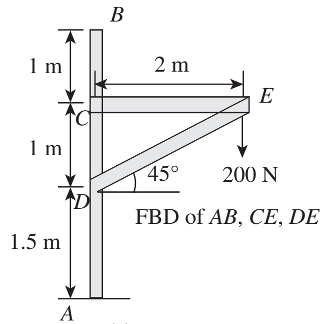


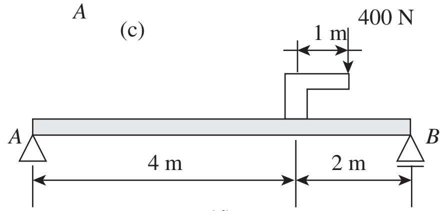
Fig. 1.101

8. Draw the free body diagrams of the bodies shown in Fig. 1.102.

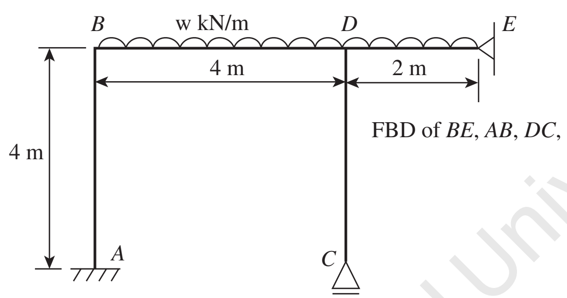




(c)



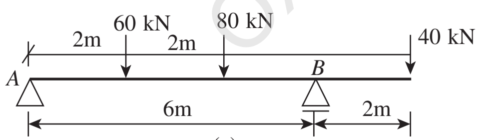
(d)



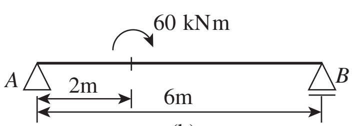
(e)

Fig. 1.102

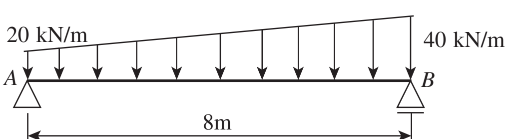
9. Determine the reactions at supports A and B for the beams loaded as shown in Fig. 1.103.



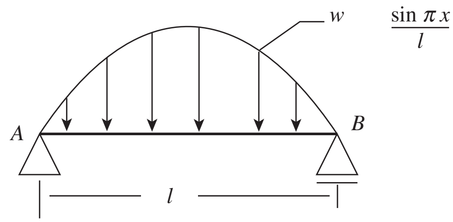
(a)



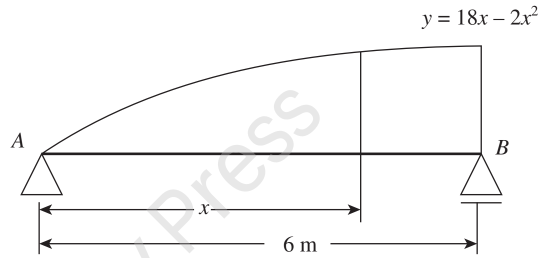
(b)



(c)



(d)



(e)

Fig. 1.103

10. For a beam loaded as shown in Fig. 1.104, if $x = l/2$, find the reactions at A and B. What is the value of x for which $R_B = 0$? What is the value of the reaction R_A in such case?

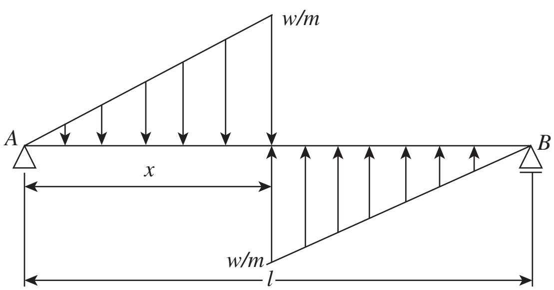


Fig. 1.104

11. Determine the reactions at supports A and B of the truss loaded as shown in Fig. 1.105.

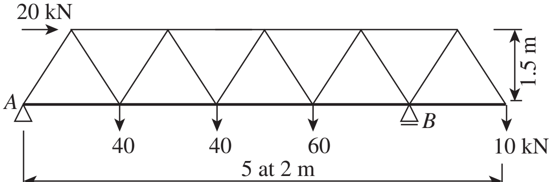


Fig. 1.105

12. Determine the reactions at A and B for the structures loaded as shown in Fig. 1.106.

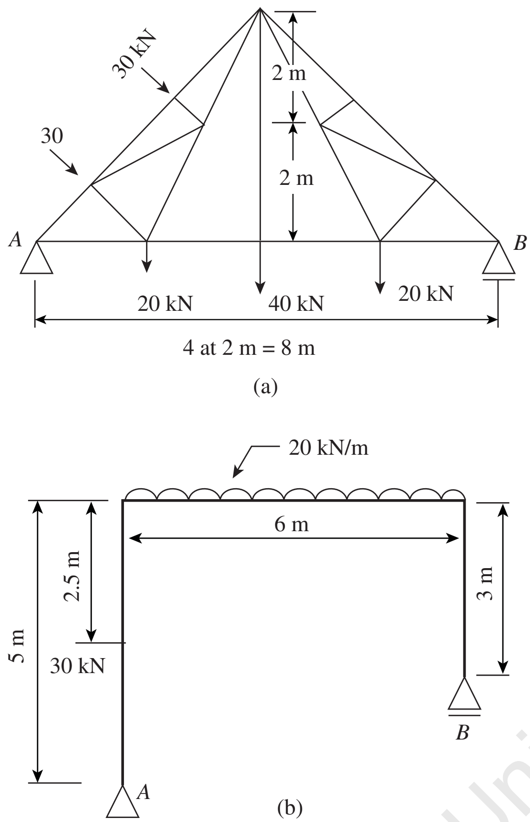


Fig. 1.106

13. Find the MI of an isoceles triangle of base 200 mm and height 300 mm about an axis through its base and a parallel axis through its centroid.
14. Find the MI of a symmetrical trapezium of height 200 mm, base 200 mm, and top face 100 mm about an axis through its centroid.
15. Find the MI about the centroidal axes of (i) a hexagon of side 150 mm and (ii) an octagon of side 200 mm.
16. Find the MI of an unequal angle section 200 × 100 × 8 about horizontal and vertical axes through its centroid. What is the minimum radius of gyration of this section?
17. Find the MI of a channel section of dimensions 200 × 80 × 10 about horizontal and vertical axes through its centroid. If two such channel sections are placed back to back, find the clear distance between the channels so that their MI about two perpendicular axes (horizontal and vertical) are equal.
18. Find the MIs of the three sections shown in Fig. 1.107 about a horizontal axis through their centroids.

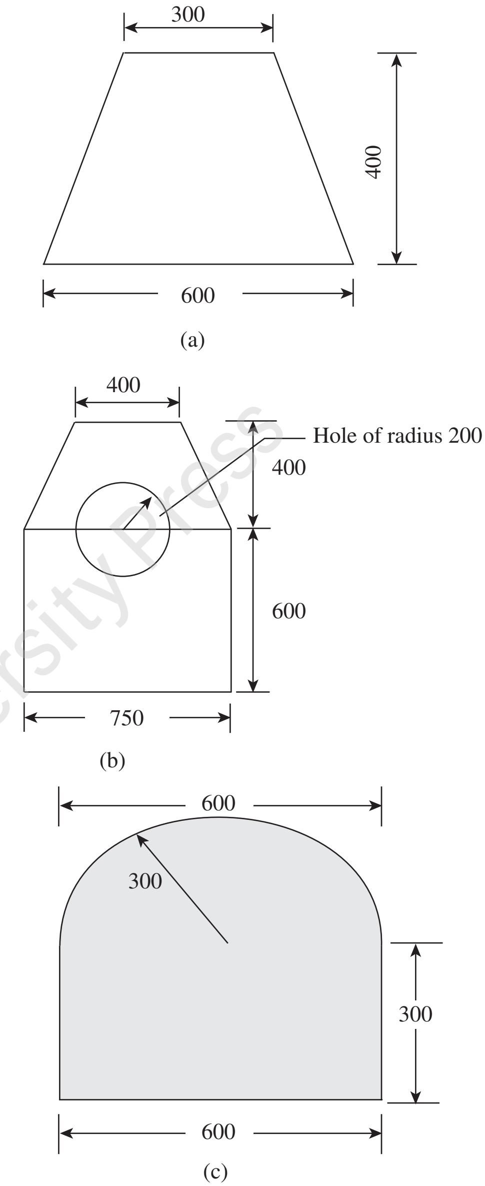


Fig. 1.107

19. Which of the following sections will have the largest radius of gyration, about an axis parallel to the base through the centroid or about a diameter, if they have the same area?
   (i) A triangle of equal sides,
   (ii) A square,
   (iii) A circle, and
   (iv) A tube of thickness 0.05 times its outer diameter.

20. Find the MI about the *X-X* and *Y-Y* axes of the areas bound by the curves shown in Fig. 1.108

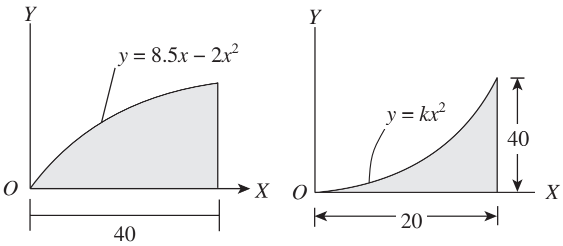


Fig. 1.108

21. Find the product of inertia of a trapezium if one of its sides is vertical and 200 mm long and the two parallel sides at right angles to it are 100 mm and 300 mm about the axis passing through the vertical side and the 300 mm base.
22. Find the product of inertia of the two triangles shown in Fig. 1.109 about the *X-X* and *Y-Y* axes

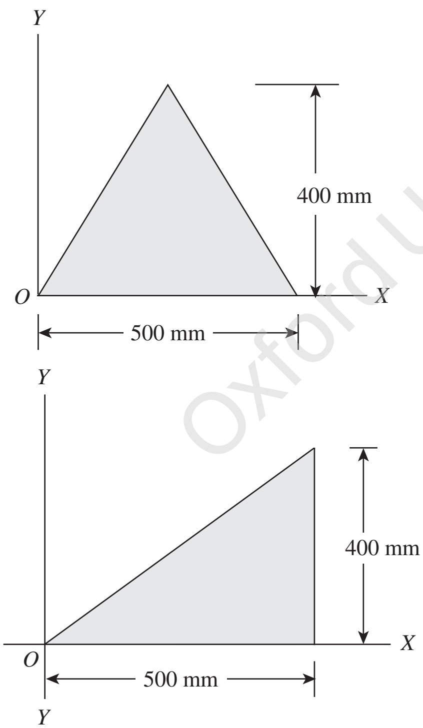


Fig. 1.109

23. Find the PI of a right-angled triangle about horizontal and vertical axes through its centroid. The base of the triangle is 250 mm long and its height is 300 mm.
24. Find the principal axes through the centroid of a right-angled triangle, of base 200 mm and height 300 mm. Find the principal MI about these axes.
25. An unequal angle section 300 × 120 × 10 is placed with its longer base horizontal. Find the principal axes through the centroid of this angle section, and principal MIs.
26. For the Z-section shown in Fig. 1.110, find the principal axes through its centroid and the principal MIs about these axes.

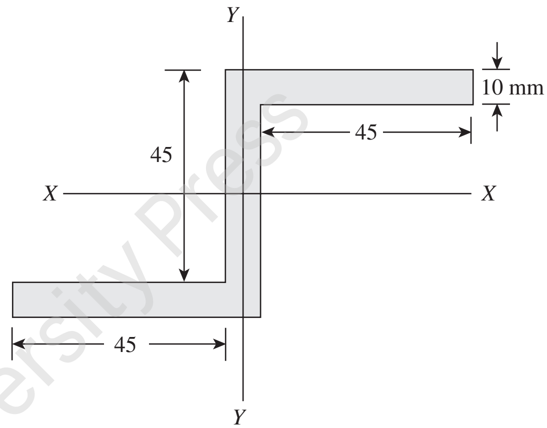
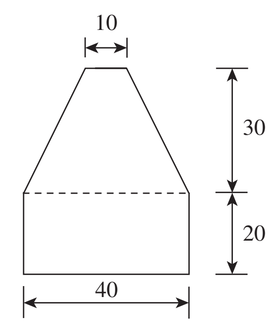


Fig. 1.110

27. For the angle section given in Problem 13, find the principal axes and inertias using Mohr's circle method.
28. For the Z-shaped section of Problem 14, find the principal axes and principal moment of inertia graphically using Mohr's circle of inertia.
29. Find the section modulus of the section shown in Fig. 1.111



(All dimensions are in mm)

Fig. 1.111

30. Determine the section modulus with respect to top and bottom fibres for the T-section shown in Fig. 1.112.

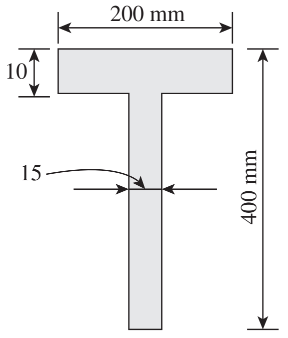


Fig. 1.112

31. Determine the section modulus of a regular hexagon of side 20 cm.
32. Compare the ratio of area to section modulus of a circular area of radius R and a hollow circular area of outer radius R and thickness 0.05 R.